AN EXPLANATION OF THE OBSERVED IRREGULARITIES IN THE MOTION OF URANUS

JOHN C. ADAMS

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OF THE

OBSERVED IRREGULARITIES

IN THE

MOTION OF URANUS,

ON THE HYPOTHESIS OF DISTURBANCES

CAUSED BY A MORE DISTANT PLANET;

WITH A DETERMINATION OF THE MASS, ORBIT, AND POSITION OF THE DISTURBING BODY.

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ADVERTISEMENT.

This Paper was communicated by the Author to the Royal Astronomical Society, and was read to that body, at their ordinary meeting, on November 13, 1846. The Press of the Society being engaged on an extensive paper, on the longitude of Valentia, by the Astronomer Royal, and it being deemed of national importance that Mr. Adams's Paper should be submitted to the world without loss of time, application was made to Capt. W. H. Smyth, R.N., President, and to the Rev. R. Sheepshanks, Secretary, of the Society, who, with their usual promptitude and zeal, granted permission for the immediate printing and publishing of the Paper by the Nautical Almanac Office; and it is under these circumstances that the investigations of Mr. Adams first appear as an extract from the Appendix to the Nautical Almanac for 1851.

W. S. STRATFORD,
Superintendent of the Nautical Almanac.

Nautical Almanac Office,
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ON THE PERTURBATIONS OF URANUS.

1. The irregularities in the motions of Uranus have for a long time engaged the attention of Astronomers. When the path of the planet became approximately known, it was found that, previously to its discovery by Sir W. Herschel in 1781, it had several times been observed as a fixed star by Flamsteed, Bradley, Mayer, and Lemonnier. Although these observations are doubtless very far inferior in accuracy to the modern ones, they must be considered valuable, in consequence of the great extension which they give to the observed arc of the planet's orbit. Bouvard, however, to whom we owe the Tables of Uranus at present in use, found that it was impossible to satisfy these observations, without attributing much larger errors to the modern observations than they admit of, and consequently founded his Tables exclusively on the latter. But in a very few years sensible errors began again to show themselves, and though the Tables were formed so recently as 1821, their error at the present time exceeds two minutes of space, and is still rapidly increasing. There appeared, therefore, no longer any sufficient reason for rejecting the ancient observations, especially since, with the exception of Flamsteed's first observation, which is more than twenty years anterior to any of the others, they are mutually confirmatory of each other.

2. Now that the discovery of another planet has confirmed in the most brilliant manner the conclusions of analysis, and enabled us with certainty to refer these irregularities to their true cause, it is unnecessary for me to enter at length upon the reasons which led me to reject the various other hypotheses which had been formed to account for them. It is sufficient to say, that they all appeared to be very improbable in themselves, and incapable of being tested by any exact calculation. Some had even supposed that at the great distance of Uranus from the Sun, the law of attraction becomes different from that of the inverse square of the distance. But the law of gravitation was too firmly established for this to be admitted, till every other hypothesis had failed, and I felt convinced that in this, as in every previous instance of the kind, the discrepancies which had for a time thrown doubts on the truth of the law, would eventually afford the most striking confirmation of it.

3. My attention was first directed to this subject several years since, by reading Mr. Airy's valuable Report on the recent progress of Astronomy. I find among my

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papers the following memorandum, dated July 3, 1841:—"Formed a design, in the beginning of this week, of investigating, as soon as possible after taking my degree, the irregularities in the motion of Uranus which are yet unaccounted for; in order to find whether they may be attributed to the action of an undiscovered planet beyond it, and if possible, thence to determine approximately the elements of its orbit, &c., which would probably lead to its discovery." Accordingly, in 1843, I attempted a first solution of the problem, assuming the orbit to be a circle, with a radius equal to twice the mean distance of Uranus from the Sun. Some assumption as to the mean distance was clearly necessary in the first instance, and Bode's law appeared to render it probable that the above would not be far from the truth. This investigation was founded exclusively on the modern observations, and the errors of the Tables were taken from those given in the Equations of Condition of Bouvard's Tables as far as the year 1821, and subsequently from the observations given in the *Astronomische Nachrichten*, and from the Cambridge and Greenwich Observations. The result showed that a good general agreement between theory and observation might be obtained; but the larger differences occurring in years where the observations used were deficient in number, and the Greenwich Planetary Observations being then in process of reduction, I applied to Mr. Airy, through the kind intervention of Professor Challis, for the observations of some years in which the agreement appeared least satisfactory. The Astronomer Royal, in the kindest possible manner, sent me, in February 1844, the results of all the Greenwich Observations of Uranus.

4. Meanwhile the Royal Academy of Sciences of Göttingen had proposed the Theory of Uranus as the subject of their mathematical prize, and although the little time which I could spare from important duties in my college prevented me from attempting the complete examination of the theory, which a competition for the prize would have required, yet this fact, together with the possession of such a valuable series of observations, induced me to undertake a new solution of the problem. I now took into account the most important terms depending on the first power of the eccentricity of the disturbing planet, retaining the same assumption as before with respect to the mean distance. For the modern observations, the errors of the Tables were taken exclusively from the Greenwich Observations as far as the year 1830, with the exception of an observation by Bessel, in 1823; and subsequently from the Cambridge and Greenwich Observations, and those given in various numbers of the *Astronomische Nachrichten.* The errors of the Tables for the ancient Observations were taken from those given in the Equations of Condition of Bouvard's Tables. After obtaining several solutions differing little from each other, by gradually taking into account more and more terms of the series expressing the Perturbations, I communicated to Professor Challis, in September 1845, the final values which I had obtained for the mass, heliocentric longitude, and elements of the orbit of the assumed planet. The same results, slightly corrected, I communicated in the following month to the Astronomer Royal. The eccentricity coming out much larger than was probable, and later observations showing that the theory founded on the first hypothesis as to the mean distance, was still sensibly in error, I afterwards repeated my investigation, supposing the mean distance to be about $\frac{1}{30}$th part less than before. The result,

which I communicated to Mr. Airy, in the beginning of September of the present year, appeared more satisfactory than my former one, the eccentricity being smaller, and the errors of theory, compared with late observations, being less, and led me to infer that the distance should be still further diminished.

5. In November, 1845, M. Le Verrier presented to the Royal Academy of Sciences at Paris, a very complete and elaborate investigation of the Theory of Uranus, as disturbed by the action of Jupiter and Saturn, in which he pointed out several small inequalities which had previously been neglected; and in June, of the present year, he followed up this investigation by a memoir, in which he attributed the residual disturbances to the action of another planet at a distance from the Sun equal to twice that of Uranus, and found a longitude for the new planet agreeing very nearly with the result which I had obtained on the same hypothesis. On the 31st of August he presented to the Academy a more complete investigation, in which he determined the mass and the elements of the orbit of the new planet, and also obtained limiting values of the mean distance and heliocentric longitude. I mention these dates merely to show that my results were arrived at independently, and previously to the publication of those of M. Le Verrier, and not with the intention of interfering with his just claims to the honours of the discovery; for there is no doubt that his researches were first published to the world, and led to the actual discovery of the planet by Dr. Galle, so that the facts stated above cannot detract, in the slightest degree, from the credit due to M. Le Verrier.

6. In order not to have an inconvenient number of equations of condition, I divided the modern observations into groups, each including a period of three years, and as Mr. Airy had shown that the error of the Tabular Radius Vector was sometimes considerable, I either selected those observations which were made near opposition, or combined the others in such a manner that the result should be nearly free from the effects of this error. From the observations of each group, the error of the Tables in heliocentric longitude was found, corresponding to the time of mean opposition in the middle year of the group. Thus were formed 21 normal errors of the Tables, corresponding to as many equidistant periods between 1780 and 1840. The error for 1780 was found by interpolating between the errors of 1781, 1782, and 1783, and those given by the Ancient Observations of 1769 and 1771, and though not entitled to the same weight as the others, cannot, I think, be liable to much uncertainty. In my last calculations, I might have used more recent observations, but in order to obtain the effect due to the change of mean distance, it was necessary that the investigation should be founded on the same elements as before, and the later observations might be used as a test of the theory.

7. In order to satisfy myself that there was no important error in Bouvard's Tables, I recomputed all the principal inequalities produced by the action of Jupiter and Saturn, and found no difference of any consequence except in the equation depending on the mean longitude of Saturn minus twice that of Uranus, the error of which had been already pointed out by Bessel. The principal equation depending on the action of Jupiter, also required correction in consequence of the increased value which has been lately obtained for the mass of that planet. The corrections to be applied to Bouvard's Tables on these accounts, are the following:

$$+1''\cdot918 \sin \{\phi_1-2\phi_2-13^\circ\ 1'\cdot5\}$$
$$+1\cdot085 \sin \{\phi-\phi_2\}$$

ϕ, ϕ_1, ϕ_2 being the mean longitudes of Jupiter, Saturn, and Uranus, respectively. In the Reduction of the Greenwich Observations, the latter correction was already taken into account. M. Hansen having also found some new inequalities in the motion of Uranus, depending on the square of the disturbing force, I re-computed the values of these, following the same method as that given by M. Delaunay in the *Conn. des Temps* for 1845, and my results agreed very closely with his, the terms to be added to the longitude being

$$+32''\cdot00 \sin \{3\phi_2-6\phi_1+2\phi+22^\circ\ 18'\cdot8\}$$
$$-\ 8\cdot35 \sin \{2\phi_2-6\phi_1+2\phi+39\ \ 10\cdot5\}$$
$$-\ 1\cdot49 \sin \{4\phi_2-6\phi_1+2\phi+34\ \ 48\cdot4\}$$

With respect to the inequalities of higher orders neglected by Bouvard, I considered that the most important of them would be, either those of long period, or those whose period was nearly equal to that of Uranus. During three-fourths of a revolution of the planet, the effects of the former class would be nearly confounded with those arising from a change in the epoch and mean motion, and those of the latter class with the effects produced by a constant change in the eccentricity and longitude of the Perihelion. The position of the planet to be determined would therefore be little affected by these terms, and the others would probably be much smaller than those which would necessarily be neglected in a first approximation to the perturbations produced by the new planet.

8. Taking into account the several corrections above-mentioned, the residual differences between the theoretical and observed heliocentric longitudes were the following:

Ancient Observations.

Year.	Observation—Theory.
1690	+61″·2
1712	+92·7
1715	+73·8
1750	−47·6
1753	−39·5
1756	−45·7
1764	−34·9
1769	−19·3
1771	− 2·3

Modern Observations.

Year.	Observation—Theory.
1780	+ 3″·46
1783	+ 8·45
1786	+12·36
1789	+19·02
1792	+18·70
1795	+21·38
1798	+20·95
1801	+22·21
1804	+24·16
1807	+22·07
1810	+23·16
1813	+22·00
1816	+22·88
1819	+20·69
1822	+20·97
1825	+18·16
1828	+10·82
1831	− 3·98
1834	−20·80
1837	−42·66
1840	−66·64

9. It is easily seen that the series expressing the correction of the *Mean* longitude in terms of the corrections applied to the elements of the orbit, is more convergent than that which gives the correction of the *true* longitude, and the same thing is true for the perturbations of the mean longitude, as compared with those of the true. The corrections found above were accordingly converted into corrections of mean longitude by multiplying each of them by the factor $\frac{r^2}{ab}$, r being the Rad. Vector, and a and b the semi-axes of the orbit. Hence these latter corrections were found to be the following:

Ancient Observations.

Year.	Observation—Theory.
1690	+62″·6
1712	+84·5
1715	+67·2
1750	—51·8
1753	—43·2
1756	—50·1
1764	—37·8
1769	—20·5
1771	— 2·4

Modern Observations.

Year.	Observation—Theory.
1780	+ 3″·42
1783	+ 8·19
1786	+11·74
1789	+17·75
1792	+17·22
1795	+19·52
1798	+19·06
1801	+20·24
1804	+22·19
1807	+20·52
1810	+21·89
1813	+21·19
1816	+22·50
1819	+20·78
1822	+21·50
1825	+18·97
1828	+11·50
1831	— 4·29
1834	—22·63
1837	—46·70
1840	—73·09

These numbers form the basis of the subsequent investigations.

10. Let $\delta\varepsilon$, δa, δe, and $\delta\varpi$ denote the corrections to be applied to the Tabular Elements of Uranus, then the correction of the mean longitude at any time t is

$$= \delta\varepsilon + 2e^2\delta\varpi + t\,\delta n - \left\{2\cos(nt+\varepsilon-\varpi) + \frac{e}{2}\cos 2(nt+\varepsilon-\varpi)\right\} e\,\delta\varpi$$
$$+ \left\{2\sin(nt+\varepsilon-\varpi) + \frac{e}{2}\sin 2(nt+\varepsilon-\varpi)\right\}\delta e$$

If we include the small term $2e^2\delta\varpi$ in the quantity $\delta\varepsilon$, this correction may be put under the following form:

$$\delta\varepsilon + t\,\delta n + \cos nt\,\delta x_1 + \sin nt\,\delta y_1 + \cos 2nt\,\delta x_2 + \sin 2nt\,\delta y_2$$

in which expression

$$\delta x_2 = \frac{1}{4} e\left\{\cos(\varepsilon-\varpi)\,\delta x_1 + \sin(\varepsilon-\varpi)\,\delta y_1\right\}$$
$$\delta y_2 = -\frac{1}{4} e\left\{\sin(\varepsilon-\varpi)\,\delta x_1 - \cos(\varepsilon-\varpi)\,\delta y_1\right\}$$

11. Also, adopting the notation of Pontécoulant's "Théorie Analytique," the perturbations of mean longitude

$$= \frac{m'}{2} \Sigma \mathrm{F}_i \sin i\,(n t - n' t + \varepsilon - \varepsilon')$$
$$+ m' e\, \Sigma\, \mathrm{G}_i \sin \{ i\,(n t - n' t + \varepsilon - \varepsilon') - (n t + \varepsilon - \varpi)\}$$
$$+ m' e'\, \Sigma\, \mathrm{H}_i \sin \{ i\,(n t - n' t + \varepsilon - \varepsilon') - (n t + \varepsilon - \varpi')\}$$

Where the accented letters belong to the disturbing planet, i takes all integral values, positive and negative, except zero, and if we put $i\,(n - n') = z$, the values of F_i G_i and H_i are the following:

$$\mathrm{F}_i = \left\{ \frac{3\, i\, n^4}{z^2\,(z^2 - n^2)} + \frac{i\, n^2}{z^2 - n^2} \right\} a\, \mathrm{A}_i + \frac{2\, n^3}{z\,(z^2 - n^2)}\, a^2 \frac{d\, \mathrm{A}_i}{d\, a}$$

$$\mathrm{G}_i = \left\{ -\frac{3\, i\,(i-1)\, n^4}{(z-n)^2\, z\,(z - 2\, n)} - \frac{i\,(i+1)\, n^2}{z\,(z - 2\, n)} + \frac{i\, n^3}{z^2 - n^2} + \frac{3\, i\, n^3}{z\,(z-n)\,(z - 2\, n)} \right\} a\, \mathrm{A}_i$$
$$+ \left\{ -\frac{3}{2} \frac{(i-1)\, n^4}{(z-n)^2\, z\,(z - 2\, n)} - \frac{1}{2} \frac{(i-1) n^2}{z\,(z - 2\, n)} - \frac{1}{2} \frac{n^2}{z^2 - n^2} - \frac{2\, i\, n^3}{z\,(z-n)\,(z - 2\, n)} \right\} a^2 \frac{d\, \mathrm{A}_i}{d\, a}$$
$$- \frac{n^3}{z\,(z-n)\,(z - 2\, n)}\, a^3 \frac{d^2\, \mathrm{A}_i}{d\, a^2}.$$

$$\mathrm{H}_i = \left\{ \frac{3}{2} \frac{(i-1)\,(2\, i - 1)\, n^4}{(z-n)^2\, z\,(z - 2 n)} + \frac{1}{2} \frac{(i-1)\,(2\, i - 1)\, n^2}{z\,(z - 2\, n)} \right\} a\, \mathrm{A}_{i-1}$$
$$+ \left\{ \frac{3}{2} \frac{(i-1)\, n^4}{(z-n)^2\, z\,(z - 2\, n)} + \frac{1}{2} \frac{(i-1)\, n^2}{z\,(z - 2\, n)} + \frac{2\, i\, n^3}{z\,(z-n)\,(z - 2\, n)} \right\} a^2 \frac{d\, \mathrm{A}_{i-1}}{d\, a}$$
$$+ \frac{n^3}{z\,(z-n)\,(z - 2\, n)}\, a^3 \frac{d^2\, \mathrm{A}_{i-1}}{d\, a^2}$$

12. Now, if we assume $\frac{a}{a'}$ or $\alpha = \sin\ 30^0 = 0 \cdot 5$, the values of the fundamental quantities b, $\alpha \frac{db}{d\alpha}$, $\alpha^2 \frac{d^2 b}{d\alpha^2}$, will be

$\log.\ b_0 = 0 \cdot 33170$; $\log.\ \alpha \frac{db_0}{d\alpha} = 9 \cdot 53765$ $\log.\ \alpha^2 \frac{d^2 b_0}{d\alpha^2} = 9 \cdot 77848$

$\log.\ b_1 = 9 \cdot 74497$; $\log.\ \alpha \frac{db_1}{d\alpha} = 9 \cdot 83868$; $\log.\ \alpha^2 \frac{d^2 b_1}{d\alpha^2} = 9 \cdot 70857$

$\log.\ b_2 = 9 \cdot 32425$; $\log.\ \alpha \frac{db_2}{d\alpha} = 9 \cdot 68012$ $\log.\ \alpha^2 \frac{d^2 b_2}{d\alpha^2} = 9 \cdot 87776$

$\log.\ b_3 = 8 \cdot 94670$; $\log.\ \alpha \frac{db_3}{d\alpha} = 9 \cdot 46315$ $\log.\ \alpha^2 \frac{d^2 b_3}{d\alpha^2} = 9 \cdot 86253$

Hence the principal inequalities of mean longitude, produced by the action of a planet whose mass is $\frac{m'}{5000}$, that of the Sun being unity, and the eccentricity of whose orbit is $\frac{e'}{20}$ will be the following:

$$
\begin{array}{lll}
-36''\cdot 99\, m' & \sin & \{nt-n't+\varepsilon-\varepsilon'\} \\
+58\cdot 97\, m' & \sin 2 & \{nt-n't+\varepsilon-\varepsilon'\} \\
+\ 5\cdot 80\, m' & \sin 3 & \{nt-n't+\varepsilon-\varepsilon'\} \\
+\ 2\cdot 06\, m' & \sin & \{n't+\varepsilon'-\varpi\} \\
-\ 4\cdot 30\, m'e' & \sin & \{n't+\varepsilon'-\varpi'\} \\
+31\cdot 25\, m' & \sin & \{nt-2n't+\varepsilon-2\varepsilon'+\varpi\} \\
-12\cdot 14\, m'e' & \sin & \{nt-2n't+\varepsilon-2\varepsilon'+\varpi'\} \\
+48\cdot 55\, m' & \sin & \{2nt-3n't+2\varepsilon-3\varepsilon'+\varpi\} \\
-93\cdot 01\, m'e' & \sin & \{2nt-3n't+2\varepsilon-3\varepsilon'+\varpi'\}
\end{array}
$$

To these may be added the following, which are of two dimensions in terms of the eccentricities:

$$
\begin{array}{lll}
+0''\cdot 57\, m' & \sin 3 & \{nt-n't+\varepsilon-\varepsilon'\} \\
-1\cdot 08\, m'e' & \sin & \{3(nt-n't+\varepsilon-\varepsilon')-\varpi+\varpi'\}
\end{array}
$$

These expressions may be put under the following form:

$$
\begin{array}{l}
h_1\cos(n-n')t+h_2\cos 2(n-n')t+h_3\cos 3(n-n')t \\
+k_1\sin(n-n')t+k_2\sin 2(n-n')t+k_3\sin 3(n-n')t \\
+p_1\cos n't+p_2\cos(n-2n')t+p_3\cos(2n-3n')t \\
+q_1\sin n't+q_2\sin(n-2n')t+q_3\sin(2n-3n')t
\end{array}
$$

13. Let the time of the mean opposition in 1810 be taken as the epoch from which t is reckoned; this date, expressed in decimal parts of a year, will be $1810\cdot 328$. Also, let 3 synodic periods of Uranus, $=3\cdot 0362$ years, be taken for the unit of time; then the change of the mean anomaly in an unit of time will be $13^\circ\ 0'\cdot 5$; also $n=13^\circ\ 0'\cdot 6$, $n'=4^\circ\ 36'\cdot 0$ $\therefore$ $n-n'=8^\circ\ 24'\cdot 6$, $n-2n'=3^\circ\ 48'\cdot 6$, $2n-3n'=12^\circ\ 13'\cdot 2$. Hence the equations of condition given by the modern observations will be of the form

$$
\begin{aligned}
c'' = {} & \delta\varepsilon+\delta x_1\cos\{13^\circ\ 0'\cdot 5\}t+\delta x_2\cos\{26^\circ\ 1'\cdot 0\}t \\
& +t\delta n+\delta y_1\sin\{13\ \ 0\cdot 5\}t+\delta y_2\sin\{26\ \ 1\cdot 0\}t \\
& +h_1\cos\{8^\circ\ 24'\cdot 6\}t+h_2\cos\{16^\circ\ 49'\cdot 2\}t+h_3\cos\{25^\circ\ 13'\cdot 8\}t \\
& +k_1\sin\{8\ \ 24\cdot 6\}t+k_2\sin\{16\ \ 49\cdot 2\}t+k_3\sin\{25\ \ 13\cdot 8\}t \\
& +p_1\cos\{4^\circ\ 36'\cdot 0\}t+p_2\cos\{3^\circ\ 48'\cdot 6\}t+p_3\cos\{12^\circ\ 13'\cdot 2\}t \\
& +q_1\sin\{4\ \ 36\cdot 0\}t+q_2\sin\{3\ \ 48\cdot 6\}t+q_3\sin\{12\ \ 13\cdot 2\}t
\end{aligned}
$$

in which t assumes all integral values from -10 to $+10$ in succession, and the several values of c'' are contained in the table given in Article 9.

14. The final equations for the corrections of the elliptic elements will be found by multiplying each equation successively by the co-efficients of $\delta\epsilon$, δn, δx_1 and δy_1, which occur in it, and adding the several results.

Let the equations be treated in a similar manner with reference to the quantities $h_1, k_1, h_2, k_2, h_3, k_3, p_2, q_2, p_3, q_3$.

It will be seen that, in consequence of the arrangement which has been given to the equations of condition, the equations thus formed naturally separate themselves into two groups, one of which involves only $\delta\epsilon$, δx_1, δx_2, with the quantities h and p, while the other involves δn, δy_1, δy_2, with the quantities k and q.

Also, the co-efficients in these equations are easily calculated by the following formulæ, putting $t=10$ in their right hand members:

$$\Sigma\, 2\cos m\,t = \frac{\sin m\,(t+\frac{1}{2})}{\sin\frac{1}{2}m}$$

$$\Sigma\, 2\,t\sin m\,t = \frac{(t+1)\sin m\,t - t\sin m\,(t+1)}{2\sin^2\frac{1}{2}m}$$

$$\Sigma\, 2\cos m\,t\cos n\,t = \frac{1}{2}\left\{\frac{\sin(m-n)\,(t+\frac{1}{2})}{\sin\frac{1}{2}(m-n)} + \frac{\sin(m+n)\,(t+\frac{1}{2})}{\sin\frac{1}{2}(m+n)}\right\}$$

$$\Sigma\, 2\sin m\,t\sin n\,t = \frac{1}{2}\left\{\frac{\sin(m-n)(t+\frac{1}{2})}{\sin\frac{1}{2}(m-n)} - \frac{\sin(m+n)\,(t+\frac{1}{2})}{\sin\frac{1}{2}(m+n)}\right\}$$

$$\Sigma\, 2\cos^2 m\,t = t+\frac{1}{2}+\frac{1}{2}\,\frac{\sin m\,(2t+1)}{\sin m}$$

$$\Sigma\, 2\sin^2 m\,t = t+\frac{1}{2}-\frac{1}{2}\,\frac{\sin m\,(2t+1)}{\sin m}$$

15. By performing the calculations, the equations of the first group are found to be the following:

$$(\epsilon)\qquad \begin{aligned} 151''{\cdot}48 = {} & 21{\cdot}0000\,\delta\epsilon + 6{\cdot}0670\,\delta x_1 - 4{\cdot}4358\,\delta x_2 \\ & +13{\cdot}6320\,h_1 + 0{\cdot}4043\,h_2 - 4{\cdot}5608\,h_3 \\ & +18{\cdot}6046\,p_1 + 19{\cdot}3384\,p_2 + 7{\cdot}3721\,p_3 \end{aligned}$$

$$(x)\qquad \begin{aligned} 246{\cdot}48 = {} & 6{\cdot}0670\,\delta\epsilon + 8{\cdot}2821\,\delta x_1 + 4{\cdot}1762\,\delta x_2 \\ & + 7{\cdot}4041\,h_1 + 8{\cdot}2523\,h_2 + 4{\cdot}6963\,h_3 \\ & + 6{\cdot}5389\,p_1 + 6{\cdot}3978\,p_2 + 8{\cdot}1831\,p_3 \end{aligned}$$

$$(h_1)\qquad \begin{aligned} 209{\cdot}74 = {} & 13{\cdot}6320\,\delta\epsilon + 7{\cdot}4041\,\delta x_1 - 0{\cdot}2337\,\delta x_2 \\ & +10{\cdot}7022\,h_1 + 4{\cdot}5356\,h_2 - 0{\cdot}0018\,h_3 \\ & +12{\cdot}7013\,p_1 + 12{\cdot}9883\,p_2 + 8{\cdot}0038\,p_3 \end{aligned}$$

$$(h_2)\qquad \begin{aligned} 242{\cdot}68 = {} & 0{\cdot}4043\,\delta\epsilon + 8{\cdot}2523\,\delta x_1 + 7{\cdot}5650\,\delta x_2 \\ & + 4{\cdot}5356\,h_1 + 10{\cdot}2960\,h_2 + 8{\cdot}1944\,h_3 \\ & + 1{\cdot}7866\,p_1 + 1{\cdot}3667\,p_2 + 7{\cdot}6671\,p_3 \end{aligned}$$

$$(h_3)\qquad \begin{aligned} 86{\cdot}67 = {} & - 4{\cdot}5608\,\delta\epsilon + 4{\cdot}6963\,\delta x_1 + 10{\cdot}5023\,\delta x_2 \\ & - 0{\cdot}0018\,h_1 + 8{\cdot}1944\,h_2 + 10{\cdot}7071\,h_3 \\ & - 3{\cdot}0812\,p_1 - 3{\cdot}5347\,p_2 + 3{\cdot}8855\,p_3 \end{aligned}$$

$$(p_2) \quad 165''\cdot99 = 19\cdot3384\,\delta\epsilon + 6\cdot3978\,\delta x_1 - 3\cdot4948\,\delta x_2 + 12\cdot9883\,h_1 + 1\cdot3667\,h_2 - 3\cdot5347\,h_3 + 17\cdot2795\,p_1 + 17\cdot9106\,p_2 + 7\cdot5423\,p_3$$

$$(p_3) \quad 242\cdot56 = 7\cdot3721\,\delta\epsilon + 8\cdot1831\,\delta x_1 + 3\cdot4071\,\delta x_2 + 8\cdot0038\,h_1 + 7\cdot6671\,h_2 + 3\cdot8855\,h_3 + 7\cdot6127\,p_1 + 7\cdot5423\,p_2 + 8\cdot2019\,p_3$$

16. By means of (ϵ) eliminate $\delta\epsilon$ from each of the other equations, and these latter become

$$(x) \quad 202''\cdot72 = 6\cdot5294\,\delta x_1 + 5\cdot4577\,\delta x_2 + 3\cdot4658\,h_1 + 8\cdot1355\,h_2 + 6\cdot0139\,h_3 + 1\cdot1640\,p_1 + 0\cdot8109\,p_2 + 6\cdot0533\,p_3$$

$$(h_1) \quad 111\cdot41 = 3\cdot4658\,\delta x_1 + 2\cdot6458\,\delta x_2 + 1\cdot8531\,h_1 + 4\cdot2731\,h_2 + 2\cdot9588\,h_3 + 0\cdot6243\,p_1 + 0\cdot4349\,p_2 + 3\cdot2183\,p_3$$

$$(h_2) \quad 239\cdot76 = 8\cdot1355\,\delta x_1 + 7\cdot6504\,\delta x_2 + 4\cdot2731\,h_1 + 10\cdot2882\,h_2 + 8\cdot2822\,h_3 + 1\cdot4284\,p_1 + 0\cdot9944\,p_2 + 7\cdot5252\,p_3$$

$$(h_3) \quad 119\cdot57 = 6\cdot0139\,\delta x_1 + 9\cdot5389\,\delta x_2 + 2\cdot9588\,h_1 + 8\cdot2822\,h_2 + 9\cdot7166\,h_3 + 0\cdot9593\,p_1 + 0\cdot6652\,p_2 + 5\cdot4866\,p_3$$

$$(p_2) \quad 26\cdot50 = 0\cdot8109\,\delta x_1 + 0\cdot5900\,\delta x_2 + 0\cdot4349\,h_1 + 0\cdot9944\,h_2 + 0\cdot6652\,h_3 + 0\cdot1470\,p_1 + 0\cdot1024\,p_2 + 0\cdot7535\,p_3$$

$$(p_3) \quad 189\cdot38 = 6\cdot0533\,\delta x_1 + 4\cdot9643\,\delta x_2 + 3\cdot2183\,h_1 + 7\cdot5252\,h_2 + 5\cdot4866\,h_3 + 1\cdot0815\,p_1 + 0\cdot7535\,p_2 + 5\cdot6139\,p_3$$

17. Again, by means of (x) eliminate δx_1 from each of the other equations, and we find

$$(h_1) \quad 3''\cdot807 = -0\cdot2512\,\delta x_2 + 0\cdot0135\,h_1 - 0\cdot0452\,h_2 - 0\cdot2334\,h_3 + 0\cdot0065\,p_1 + 0\cdot0045\,p_2 + 0\cdot0052\,p_3$$

$$(h_2) \quad -12\cdot821 = 0\cdot8502\,\delta x_2 - 0\cdot0452\,h_1 + 0\cdot1515\,h_2 + 0\cdot7890\,h_3 - 0\cdot0219\,p_1 - 0\cdot0160\,p_2 - 0\cdot0171\,p_3$$

$$(h_3) \quad -67\cdot149 = 4\cdot5120\,\delta x_2 - 0\cdot2334\,h_1 + 0\cdot7890\,h_2 + 4\cdot1775\,h_3 - 0\cdot1128\,p_1 - 0\cdot0817\,p_2 - 0\cdot0888\,p_3$$

$$(p_2) \quad 1\cdot327 = -0\cdot0878\,\delta x_2 + 0\cdot0045\,h_1 - 0\cdot0160\,h_2 - 0\cdot0817\,h_3 + 0\cdot0024\,p_1 + 0\cdot0017\,p_2 + 0\cdot0018\,p_3$$

$$(p_3) \quad 1\cdot448 = -0\cdot0955\,\delta x_2 + 0\cdot0052\,h_1 - 0\cdot0171\,h_2 - 0\cdot0888\,h_3 + 0\cdot0024\,p_1 + 0\cdot0018\,p_2 + 0\cdot0020\,p_3$$

18. Similarly, the equations of the second group are found to be

$$(n) \quad -171''\cdot27 = 77\cdot0000\,\delta n + 9\cdot3938\,\delta y_1 - 1\cdot2183\,\delta y_2 + 8\cdot8463\,k_1 + 7\cdot3034\,k_2 - 0\cdot5927\,k_3 + 5\cdot7519\,q_1 + 4\cdot8755\,q_2 + 9\cdot5583\,q_3$$

$$(y) \quad -166\cdot33 = 93\cdot9380\,\delta n + 12\cdot7179\,\delta y_1 + 1\cdot8907\,\delta y_2 + 11\cdot2022\,k_1 + 11\cdot0848\,k_2 + 2\cdot6731\,k_3 + 7\cdot0956\,q_1 + 5\cdot9913\,q_2 + 12\cdot7441\,q_3$$

$$
\begin{aligned}
(k_1)\quad -182''.87 = {} & 88.4630\,\delta n + 11.2022\,\delta y_1 - 0.3210\,\delta y_2 \\
& + 10.2978\,k_1 + 9.0964\,k_2 + 0.4061\,k_3 \\
& + 6.6370\,q_1 + 5.6163\,q_2 + 11.3346\,q_3 \\
(k_2)\quad -89.07 = {} & 73.0340\,\delta n + 11.0848\,\delta y_1 + 4.8266\,\delta y_2 \\
& + 9.0964\,k_1 + 10.7040\,k_2 + 5.4376\,k_3 \\
& + 5.5855\,q_1 + 4.6976\,q_2 + 10.9375\,q_3 \\
(k_3)\quad +124.80 = {} & -5.9270\,\delta n + 2.6731\,\delta y_1 + 10.4253\,\delta y_2 \\
& + 0.4061\,k_1 + 5.4376\,k_2 + 10.2929\,k_3 \\
& - 0.2497\,q_1 - 0.2643\,q_2 + 2.1788\,q_3 \\
(q_2)\quad -107.02 = {} & 48.7550\,\delta n + 5.9913\,\delta y_1 - 0.6614\,\delta y_2 \\
& + 5.6163\,k_1 + 4.6976\,k_2 - 0.2643\,k_3 \\
& + 3.6475\,q_1 + 3.0894\,q_2 + 6.0897\,q_3 \\
(q_3)\quad -175.89 = {} & 95.5830\,\delta n + 12.7441\,\delta y_1 + 1.3845\,\delta y_2 \\
& + 11.3346\,k_1 + 10.9375\,k_2 + 2.1788\,k_3 \\
& + 7.2084\,q_1 + 6.0897\,q_2 + 12.7981\,q_3
\end{aligned}
$$

19. By means of (n), eliminate δn from each of the other equations, and we have

$$
\begin{aligned}
(y)\quad 42''.61 = {} & 1.2578\,\delta y_1 + 3.3771\,\delta y_2 + 0.4100\,k_1 + 2.1748\,k_2 \\
& + 3.3962\,k_3 + 0.0785\,q_1 + 0.0433\,q_2 + 1.0833\,q_3 \\
(k_1)\quad 13.90 = {} & 0.4100\,\delta y_1 + 1.0787\,\delta y_2 + 0.1346\,k_1 + 0.7057\,k_2 \\
& + 1.0871\,k_3 + 0.0288\,q_1 + 0.0150\,q_2 + 0.3534\,q_3 \\
(k_2)\quad 73.38 = {} & 2.1748\,\delta y_1 + 5.9822\,\delta y_2 + 0.7057\,k_1 + 3.7767\,k_2 \\
& + 5.9998\,k_3 + 0.1298\,q_1 + 0.0732\,q_2 + 1.8715\,q_3 \\
(k_3)\quad 111.62 = {} & 3.3962\,\delta y_1 + 10.3315\,\delta y_2 + 1.0871\,k_1 + 5.9998\,k_2 \\
& + 10.2473\,k_3 + 0.1930\,q_1 + 0.1110\,q_2 + 2.9145\,q_3 \\
(q_2)\quad 1.42 = {} & 0.0433\,\delta y_1 + 0.1100\,\delta y_2 + 0.0150\,k_1 + 0.0732\,k_2 \\
& + 0.1110\,k_3 + 0.0055\,q_1 + 0.0023\,q_2 + 0.0375\,q_3 \\
(q_3)\quad 36.72 = {} & 1.0833\,\delta y_1 + 2.8969\,\delta y_2 + 0.3534\,k_1 + 1.8715\,k_2 \\
& + 2.9145\,k_3 + 0.0684\,q_1 + 0.0375\,q_2 + 0.9330\,q_3
\end{aligned}
$$

20. Again, eliminating δy_1 by means of (y) we find

$$
\begin{aligned}
(k_1)\quad 0''.009 = {} & -0.0221\,\delta y_2 + 0.0010\,k_1 - 0.0032\,k_2 - 0.0200\,k_3 \\
& + 0.0032\,q_1 + 0.0009\,q_2 + 0.0003\,q_3 \\
(k_2)\quad -0.301 = {} & 0.1430\,\delta y_2 - 0.0032\,k_1 + 0.0162\,k_2 + 0.1274\,k_3 \\
& - 0.0059\,q_1 - 0.0017\,q_2 - 0.0016\,q_3 \\
(k_3)\quad -3.443 = {} & 1.2129\,\delta y_2 - 0.0200\,k_1 + 0.1274\,k_2 + 1.0769\,k_3 \\
& - 0.0189\,q_1 - 0.0059\,q_2 - 0.0105\,q_3 \\
(q_2)\quad -0.045 = {} & -0.0062\,\delta y_2 + 0.0009\,k_1 - 0.0017\,k_2 - 0.0059\,k_3 \\
& + 0.0028\,q_1 + 0.0008\,q_2 + 0.0002\,q_3 \\
(q_3)\quad +0.017 = {} & -0.0116\,\delta y_2 + 0.0003\,k_1 - 0.0016\,k_2 - 0.0105\,k_3 \\
& + 0.0008\,q_1 + 0.0002\,q_2 + 0.0000\,q_3
\end{aligned}
$$

21. From the equations remaining in the two groups after the elimination of $\delta\epsilon$, δn, δx_1, δy_1 it will be easy, when approximate values of the mass and mean longitude of the disturbing planet have been found, to deduce the final equations for determining these quantities more accurately by the method of minimum squares.

It may be observed, however, that the equations in each group are very nearly identical with each other, aud therefore two final equations may be formed by simply adding together the several equations of each group, after giving the unknown quantities the same sign in them all. Thus we find

$$\begin{aligned}86''\cdot552 &= -5\cdot7967\,\delta x_2+0\cdot3018\,h_1-1\cdot0188\,h_2-5\cdot3704\,h_3\\ &\quad+0\cdot1460\,p_1+0\cdot1056\,p_2+0\cdot1149\,p_3\\ 3\cdot725 &= -1\cdot3958\,\delta y_2+0\cdot0254\,k_1-0\cdot1501\,k_2-1\cdot2407\,k_3\\ &\quad+0\cdot0316\,q_1+0\cdot0095\,q_2+0\cdot0127\,q_3\end{aligned}$$

22. If in the expressions before given for δx_2 and δy_2 we substitute $e=0\cdot046679$ and $\epsilon-\varpi=50°15'\cdot8$, we obtain

$$\begin{aligned}\delta x_2 &= 0\cdot007460\,\delta x_1+0\cdot008974\,\delta y_1\\ \delta y_2 &= -0\cdot008974\,\delta x_1+0\cdot007460\,\delta y_1\end{aligned}$$

Substituting these values in the equations (x) and (y) and in those just found, it may be seen that by adding to the latter equations

$$0\cdot006768\,(x)+0\cdot040287\,(y)$$

and $\quad -0\cdot001869\,(x)+0\cdot008187\,(y)$ respectively,

δx_1 and δy_1 will be eliminated, and we shall obtain the following equations:

$$\begin{aligned}(1)\qquad 89''\cdot641 &= 0\cdot3252\,h_1-0\cdot9637\,h_2-5\cdot3297\,h_3\\ &\quad+0\cdot0165\,k_1+0\cdot0876\,k_2+0\cdot1368\,k_3\\ &\quad+0\cdot1539\,p_1+0\cdot1111\,p_2+0\cdot1559\,p_3\\ &\quad+0\cdot0032\,q_1+0\cdot0017\,q_2+0\cdot0436\,q_3\\ (2)\qquad 3\cdot695 &= -0\cdot0065\,h_1-0\cdot0152\,h_2-0\cdot0112\,h_3\\ &\quad+0\cdot0288\,k_1-0\cdot1323\,k_2-1\cdot2129\,k_3\\ &\quad-0\cdot0022\,p_1-0\cdot0015\,p_2-0\cdot0113\,p_3\\ &\quad+0\cdot0323\,q_1+0\cdot0099\,q_2+0\cdot0215\,q_3\end{aligned}$$

23. These equations would be sufficient for determining the mass of the disturbing planet and its longitude at the epoch, if the eccentricity of the orbit were neglected. We will now proceed to find equations from the Ancient Observations for determining the eccentricity and longitude of the Perihelion.

The equations of condition given by the Ancient Observations are the following:

$$\begin{aligned}62''\cdot6 = \quad & \delta\epsilon-0\cdot8776\,\delta x_1+0\cdot5402\,\delta x_2+0\cdot8712\,h_1+0\cdot5180\,h_2\\ & -39\cdot31\,\delta n-0\cdot4795\,\delta y_1+0\cdot8415\,\delta y_2+0\cdot4909\,k_1+0\cdot8554\,k_2\\ & +0\cdot0314\,h_3-0\cdot9999\,p_1-0\cdot8640\,p_2-0\cdot5055\,p_3\\ & +0\cdot9995\,k_3+0\cdot0145\,q_1-0\cdot5035\,q_2-0\cdot8628\,q_3\end{aligned}$$

$$\begin{aligned}
84''\cdot5= &\ \delta\epsilon+0\cdot4975\ \delta x_1-0\cdot5050\ \delta x_2+0\cdot0288\ h_1-0\cdot9984\ h_2\\
&-32\cdot30\ \delta n-0\cdot8675\ \delta y_1-0\cdot8631\ \delta y_2+0\cdot9996\ k_1+0\cdot0573\ k_2\\
&-0\cdot0860\ h_3-0\cdot8534\ p_1-0\cdot5456\ p_2+0\cdot8220\ p_3\\
&-0\cdot9963\ k_3-0\cdot5213\ q_1-0\cdot8380\ q_2-0\cdot5695\ q_3
\end{aligned}$$

$$\begin{aligned}
67\cdot2= &\ \delta\epsilon+0\cdot6732\ \delta x_1-0\cdot0935\ \delta x_2-0\cdot1120\ h_1-0\cdot9749\ h_2\\
&-31\cdot34\ \delta n-0\cdot7394\ \delta y_1-0\cdot9956\ \delta y_2+0\cdot9937\ k_1-0\cdot2227\ k_2\\
&+0\cdot3305\ h_3-0\cdot8105\ p_1-0\cdot4912\ p_2+0\cdot9206\ p_3\\
&-0\cdot9138\ k_3-0\cdot5857\ q_1-0\cdot8711\ q_2-0\cdot3905\ q_3
\end{aligned}$$

$$\begin{aligned}
-51\cdot8= &\ \delta\epsilon-0\cdot2616\ \delta x_1-0\cdot8631\ \delta x_2-0\cdot9649\ h_1+0\cdot8618\ h_2\\
&-19\cdot59\ \delta n+0\cdot9652\ \delta y_1-0\cdot5050\ \delta y_2-0\cdot2627\ k_1+0\cdot5073\ k_2\\
&-0\cdot6982\ h_3-0\cdot0023\ p_1+0\cdot2650\ p_2-0\cdot5090\ p_3\\
&-0\cdot7159\ k_3-1\cdot0000\ q_1-0\cdot9642\ q_2+0\cdot8607\ q_3
\end{aligned}$$

$$\begin{aligned}
-43\cdot2= &\ \delta\epsilon-0\cdot4741\ \delta x_1-0\cdot5505\ \delta x_2-0\cdot9154\ h_1+0\cdot6758\ h_2\\
&-18\cdot58\ \delta n+0\cdot8805\ \delta y_1-0\cdot8348\ \delta y_2-0\cdot4025\ k_1+0\cdot7371\ k_2\\
&-0\cdot3220\ h_3+0\cdot0787\ p_1+0\cdot3291\ p_2-0\cdot6814\ p_3\\
&-0\cdot9467\ k_3-0\cdot9969\ q_1-0\cdot9443\ q_2+0\cdot7319\ q_3
\end{aligned}$$

$$\begin{aligned}
-50\cdot1= &\ \delta\epsilon-0\cdot6430\ \delta x_1-0\cdot1731\ \delta x_2-0\cdot8543\ h_1+0\cdot4599\ h_2\\
&-17\cdot68\ \delta n+0\cdot7659\ \delta y_1-0\cdot9849\ \delta y_2-0\cdot5198\ k_1+0\cdot8879\ k_2\\
&+0\cdot0686\ h_3+0\cdot1510\ p_1+0\cdot3848\ p_2-0\cdot8085\ p_3\\
&-0\cdot9976\ k_3-0\cdot9885\ q_1-0\cdot9230\ q_2+0\cdot5885\ q_3
\end{aligned}$$

$$\begin{aligned}
-37\cdot8= &\ \delta\epsilon-0\cdot9492\ \delta x_1+0\cdot8021\ \delta x_2-0\cdot6189\ h_1-0\cdot2340\ h_2\\
&-15\cdot25\ \delta n+0\cdot3145\ \delta y_1-0\cdot5972\ \delta y_2-0\cdot7855\ k_1+0\cdot9722\ k_2\\
&+0\cdot9085\ h_3+0\cdot3396\ p_1+0\cdot5287\ p_2-0\cdot9939\ p_3\\
&-0\cdot4179\ k_3-0\cdot9106\ q_1-0\cdot8488\ q_2+0\cdot1100\ q_3
\end{aligned}$$

$$\begin{aligned}
-20\cdot5= &\ \delta\epsilon-0\cdot9985\ \delta x_1+0\cdot9942\ \delta x_2-0\cdot4128\ h_1-0\cdot6591\ h_2\\
&-13\cdot60\ \delta n-0\cdot0538\ \delta y_1+0\cdot1074\ \delta y_2-0\cdot9108\ k_1+0\cdot7520\ k_2\\
&+0\cdot9571\ h_3+0\cdot4607\ p_1+0\cdot6182\ p_2-0\cdot9711\ p_3\\
&+0\cdot2899\ k_3-0\cdot8875\ q_1-0\cdot7860\ q_2-0\cdot2385\ q_3
\end{aligned}$$

$$\begin{aligned}
-\ 2\cdot4= &\ \delta\epsilon-0\cdot9633\ \delta x_1+0\cdot8560\ \delta x_2-0\cdot2807\ h_1-0\cdot8424\ h_2\\
&-12\cdot64\ \delta n-0\cdot2684\ \delta y_1+0\cdot5170\ \delta y_2-0\cdot9598\ k_1+0\cdot5388\ k_2\\
&+0\cdot7536\ h_3+0\cdot5279\ p_1+0\cdot6670\ p_2-0\cdot9023\ p_3\\
&+0\cdot6574\ k_3-0\cdot8493\ q_1-0\cdot7451\ q_2-0\cdot4310\ q_3
\end{aligned}$$

24. From each of these equations eliminate $\delta\epsilon$, δn, δx_1, and δy_1, by means of the equations (ϵ), (n), (x), and (y) before found, and we have the following:

$$\begin{aligned}
-142''\cdot0= &\ 1\cdot7265\ \delta x_2+0\cdot8412\ h_1+1\cdot9521\ h_2+\ 1\cdot3230\ h_3\\
&-11\cdot3691\ \delta y_2+3\cdot6001\ k_1-2\cdot8793\ k_2-10\cdot9578\ k_3\\
&-1\cdot6779\ p_1-1\cdot6400\ p_2+\ 0\cdot2249\ p_3\\
&+2\cdot6815\ q_1+1\cdot8369\ q_2+\ 0\cdot2995\ q_3
\end{aligned}$$

$$
\begin{aligned}
-105''{\cdot}2 = &- 0{\cdot}4681\,\delta x_2 - 0{\cdot}7311\,h_1 - 1{\cdot}2776\,h_2 - 0{\cdot}0609\,h_3 \\
&- 9{\cdot}6249\,\delta y_2 + 3{\cdot}7087\,k_1 - 2{\cdot}1926\,k_2 - 9{\cdot}5426\,k_3 \\
&\qquad\qquad - 1{\cdot}7765\,p_1 - 1{\cdot}4924\,p_2 + 0{\cdot}2786\,p_3 \\
&\qquad\qquad + 1{\cdot}6997\,q_1 + 1{\cdot}1014\,q_2 + 0{\cdot}7934\,q_3
\end{aligned}
$$

$$
\begin{aligned}
-126{\cdot}1 = &- 0{\cdot}2035\,\delta x_2 - 0{\cdot}9653\,h_1 - 1{\cdot}4730\,h_2 + 0{\cdot}1937\,h_3 \\
&- 9{\cdot}7719\,\delta y_2 + 3{\cdot}5895\,k_1 - 2{\cdot}5827\,k_2 - 9{\cdot}5123\,k_3 \\
&\qquad\qquad - 1{\cdot}7649\,p_1 - 1{\cdot}4598\,p_2 + 0{\cdot}2133\,p_3 \\
&\qquad\qquad + 1{\cdot}5629\,q_1 + 1{\cdot}0070\,q_2 + 0{\cdot}8437\,q_3
\end{aligned}
$$

$$
\begin{aligned}
-199{\cdot}1 = &- 0{\cdot}1917\,\delta x_2 - 1{\cdot}3218\,h_1 + 1{\cdot}5284\,h_2 + 0{\cdot}0260\,h_3 \\
&- 9{\cdot}8232\,\delta y_2 + 0{\cdot}8943\,k_1 - 3{\cdot}4359\,k_2 - 9{\cdot}9270\,k_3 \\
&\qquad\qquad - 0{\cdot}7901\,p_1 - 0{\cdot}5885\,p_2 - 0{\cdot}3497\,p_3 \\
&\qquad\qquad + 0{\cdot}2540\,q_1 + 0{\cdot}1607\,q_2 + 0{\cdot}4028\,q_3
\end{aligned}
$$

$$
\begin{aligned}
-174{\cdot}7 = &\ 0{\cdot}2985\,\delta x_2 - 1{\cdot}1595\,h_1 + 1{\cdot}6072\,h_2 + 0{\cdot}5979\,h_3 \\
&- 9{\cdot}5788\,\delta y_2 + 0{\cdot}7062\,k_1 - 2{\cdot}9425\,k_2 - 9{\cdot}5877\,k_3 \\
&\qquad\qquad - 0{\cdot}6712\,p_1 - 0{\cdot}4970\,p_2 - 0{\cdot}3251\,p_3 \\
&\qquad\qquad + 0{\cdot}1946\,q_1 + 0{\cdot}1238\,q_2 + 0{\cdot}3277\,q_3
\end{aligned}
$$

$$
\begin{aligned}
-166{\cdot}7 = &\ 0{\cdot}8171\,\delta x_2 - 1{\cdot}0088\,h_1 + 1{\cdot}6018\,h_2 + 1{\cdot}1442\,h_3 \\
&- 9{\cdot}1122\,\delta y_2 + 0{\cdot}5586\,k_1 - 2{\cdot}4890\,k_2 - 9{\cdot}0258\,k_3 \\
&\qquad\qquad - 0{\cdot}5688\,p_1 - 0{\cdot}4203\,p_2 - 0{\cdot}2956\,p_3 \\
&\qquad\qquad + 0{\cdot}1498\,q_1 + 0{\cdot}0958\,q_2 + 0{\cdot}2658\,q_3
\end{aligned}
$$

$$
\begin{aligned}
-114{\cdot}2 = &\ 2{\cdot}0482\,\delta x_2 - 0{\cdot}6027\,h_1 + 1{\cdot}2894\,h_2 + 2{\cdot}2661\,h_3 \\
&- 6{\cdot}6781\,\delta y_2 + 0{\cdot}2576\,k_1 - 1{\cdot}3421\,k_2 - 6{\cdot}4080\,k_3 \\
&\qquad\qquad - 0{\cdot}3256\,p_1 - 0{\cdot}2384\,p_2 - 0{\cdot}1971\,p_3 \\
&\qquad\qquad + 0{\cdot}0628\,q_1 + 0{\cdot}0419\,q_2 + 0{\cdot}1298\,q_3
\end{aligned}
$$

$$
\begin{aligned}
-72{\cdot}4 = &\ 2{\cdot}2815\,\delta x_2 - 0{\cdot}3786\,h_1 + 0{\cdot}9257\,k_2 + 2{\cdot}3601\,h_3 \\
&- 4{\cdot}4181\,\delta y_2 + 0{\cdot}1283\,k_1 - 0{\cdot}7339\,k_2 - 4{\cdot}1495\,k_3 \\
&\qquad\qquad - 0{\cdot}1957\,p_1 - 0{\cdot}1428\,p_2 - 0{\cdot}1286\,p_3 \\
&\qquad\qquad + 0{\cdot}0283\,q_1 + 0{\cdot}0198\,q_2 + 0{\cdot}0671\,q_3
\end{aligned}
$$

$$
\begin{aligned}
-42{\cdot}0 = &\ 2{\cdot}1139\,\delta x_2 - 0{\cdot}2652\,h_1 + 0{\cdot}6985\,h_2 + 2{\cdot}1241\,h_3 \\
&- 3{\cdot}1027\,\delta y_2 + 0{\cdot}0772\,k_1 - 0{\cdot}4646\,h_2 - 2{\cdot}8790\,k_3 \\
&\qquad\qquad - 0{\cdot}1348\,p_1 - 0{\cdot}0984\,p_2 - 0{\cdot}0924\,p_3 \\
&\qquad\qquad + 0{\cdot}0154\,q_1 + 0{\cdot}0114\,q_2 + 0{\cdot}0412\,q_3
\end{aligned}
$$

25. The largest terms depending on the eccentricity of the disturbing planet occur in p_3, q_3; it will be proper, therefore, to combine the above equations in such a manner that these quantities may acquire the largest co-efficients possible. This will be done by multiplying each equation by a quantity nearly proportional to the co-efficient of each of the unknown quantities p_3 and q_3, and adding together the several results. It was thought unsafe to employ the first of the above equations, since it is derived from the single observation of Flamsteed, made in 1690, twenty-two years anterior to any other observation.

Hence the equation for finding p_3 may be formed by multiplying the above equations, taken in order, by

$$-0\cdot8,\ -0\cdot6,\ +1\cdot0,\ +1\cdot0,\ +0\cdot9,\ +0\cdot6,\ +0\cdot4,\ +0\cdot3$$

beginning with the second ; and the equation for q_3 by multiplying the same equations by

$$1\cdot0,\quad 1\cdot0,\quad 0\cdot5,\quad 0\ 4,\quad 0\cdot3,\quad 0\cdot2,\quad 0\cdot1,\quad 0\cdot1,$$

Hence we obtain

$$\begin{aligned} -474''\cdot1 = {} & 4\cdot114\,\delta x_2 - 2\cdot817\,h_1 + 7\cdot837\,h_2 + 4\cdot528\,h_3 \\ & -20\cdot745\,\delta y_2 - 2\cdot789\,k_1 - 6\cdot551\,h_2 - 20\cdot666\,k_3 \\ & +0\cdot193\,p_1 + 0\cdot377\,p_2 - 1\cdot489\,p_3 \\ & -1\cdot660\,q_1 - 1\cdot078\,q_2 - 0\cdot054\,q_3 \end{aligned}$$

$$\begin{aligned} -485\cdot0 = {} & 0\cdot446\,\delta x_2 - 3\cdot308\,h_1 - 0\cdot442\,h_2 + 1\cdot629\,h_3 \\ & -32\cdot961\,\delta y_2 + 8\cdot267\,k_1 - 8\cdot805\,k_2 - 32\cdot546\,k_3 \\ & -4\cdot473\,p_1 - 3\cdot643\,p_2 + 0\cdot037\,p_3 \\ & +3\cdot530\,q_1 + 2\cdot278\,q_2 + 2\cdot086\,q_3 \end{aligned}$$

26. Eliminate δx_2 and δy_2 from these equations by means of (x) and (y) and they become

$$(3)\qquad \begin{aligned} -476''\cdot7 = {} & -2\cdot930\,h_1 + 7\cdot572\,h_2 + 4\cdot332\,h_3 \\ & -2\cdot751\,k_1 - 6\cdot348\,k_2 - 20\cdot350\,k_3 \\ & +0\cdot155\,p_1 + 0\cdot350\,p_2 - 1\cdot686\,p_3 \\ & -1\cdot653\,q_1 - 1\cdot074\,q_2 + 0\cdot047\,q_3 \end{aligned}$$

$$(4)\qquad \begin{aligned} -485\cdot9 = {} & -3\cdot463\,h_1 - 0\cdot805\,h_2 + 1\cdot360\,h_3 \\ & +8\cdot345\,k_1 - 8\cdot391\,k_2 - 31\cdot900\,k_3 \\ & -4\cdot525\,p_1 - 3\cdot679\,p_2 - 0\cdot233\,p_3 \\ & +3\cdot545\,q_1 + 2\cdot286\,q_2 + 2\cdot292\,q_3 \end{aligned}$$

These equations, with (1) and (2) of Article 22, suffice for the solution of our problem.

27. Eliminate the left hand members from equations (2), (3), (4), by means of equation (1) and we have

$$\begin{aligned} 0 = {} & 0\cdot4819\,h_1 - 0\cdot5950\,h_2 - 5\cdot0570\,h_3 + 0\cdot2063\,p_1 + 0\cdot1475\,p_2 + 0\cdot4300\,p_3 \\ & -0\cdot6812\,k_1 + 3\cdot2982\,k_2 + 29\cdot5618\,k_3 - 0\cdot7804\,q_1 - 0\cdot2375\,q_2 - 0\cdot4789\,q_3 \end{aligned}$$

$$\begin{aligned} 0 = {} & -1\cdot2005\,h_1 + 2\cdot4466\,h_2 - 24\cdot0122\,h_3 + 0\cdot9735\,p_1 + 0\cdot9412\,p_2 - 0\cdot8575\,p_3 \\ & -2\cdot6633\,k_1 - 5\cdot8825\,k_2 - 19\cdot6219\,h_3 - 1\cdot6362\,q_1 - 1\cdot0648\,q_2 + 0\cdot2791\,q_3 \end{aligned}$$

$$\begin{aligned} 0 = {} & -1\cdot7003\,h_1 - 6\cdot0294\,h_2 - 27\cdot5295\,h_3 - 3\cdot6908\,p_1 - 3\cdot0772\,p_2 + 0\cdot6118\,p_3 \\ & +8\cdot4344\,k_1 - 7\cdot9162\,k_2 - 31\cdot1583\,k_3 + 3\cdot5621\,q_1 + 2\cdot2954\,q_2 + 2\cdot5285\,q_3 \end{aligned}$$

28. If now we put $\varepsilon - \varepsilon' = \theta$ and $\varepsilon - \varpi = \beta$, it is easily seen that

$$\frac{h_1}{m'} = -36''\!\cdot\!99 \sin\theta, \qquad \frac{h_2}{m'} = 58''\!\cdot\!97 \sin 2\theta$$

$$\frac{k_1}{m'} = -36\cdot 99 \cos\theta, \qquad \frac{k_2}{m'} = 58\cdot 97 \cos 2\theta$$

$$\frac{h_3}{m'} = \quad 5''\!\cdot\!80 \sin 3\theta \quad +0\cdot 007460\frac{p_3}{m'} + 0\cdot 008974\,\frac{q_3}{m'}$$

$$\frac{k_3}{m'} = \quad 5\cdot 80 \cos 3\theta \quad -0\cdot 008974\frac{p_3}{m'} + 0\cdot 007460\,\frac{q_3}{m'}$$

$$\frac{p_1}{m'} = \quad 0\cdot 18 \sin(\theta - \beta) \quad -0\cdot 046247\left\{\frac{p_3}{m'}\cos 2\theta - \frac{q_3}{m'}\sin 2\theta\right\}$$

$$\frac{q_1}{m'} = -\ 0\cdot 18 \cos(\theta - \beta) \quad +0\cdot 046247\left\{\frac{p_3}{m'}\sin 2\theta + \frac{q_3}{m'}\cos 2\theta\right\}$$

$$\frac{p_2}{m'} = \ 24\cdot 91 \sin(2\theta - \beta) + 0\cdot 13055\ \left\{\frac{p_3}{m'}\cos\theta - \frac{q_3}{m'}\sin\theta\right\}$$

$$\frac{q_2}{m'} = \ 24\cdot 91 \cos(2\theta - \beta) + 0\cdot 13055\ \left\{\frac{p_3}{m'}\sin\theta + \frac{q_3}{m'}\cos\theta\right\}$$

29. Substituting these expressions in the equations of Art. 27, and putting for β its value $50^\circ\ 15'\cdot 8$, we obtain, after a slight reduction,

$$\begin{aligned}
0 = &-(1\cdot 24782)\sin\theta + (1\cdot 40248)\cos\theta - (1\cdot 57155)\sin 2\theta + (2\cdot 27388)\cos 2\theta \\
&-(1\cdot 46746)\sin 3\theta + (2\cdot 23430)\cos 3\theta + (9\cdot 10380)\frac{p_3}{m'} - (9\cdot 48254)\frac{q_3}{m'} \\
&+(8\cdot 28455)\left\{\frac{p_3}{m'}\cos\theta - \frac{q_3}{m'}\sin\theta\right\} - (8\cdot 49138)\left\{\frac{p_3}{m'}\sin\theta + \frac{q_3}{m'}\cos\theta\right\} \\
&-(7\cdot 97958)\left\{\frac{p_3}{m'}\cos 2\theta - \frac{q_3}{m'}\sin 2\theta\right\} - (8\cdot 55742)\left\{\frac{p_3}{m'}\sin 2\theta + \frac{q_3}{m'}\cos 2\theta\right\}
\end{aligned}$$

$$\begin{aligned}
0 = &\ \ (1\cdot 65083)\sin\theta + (1\cdot 99378)\cos\theta + (2\cdot 14259)\sin 2\theta - (2\cdot 58192)\cos 2\theta \\
&-(2\cdot 14400)\sin 3\theta - (2\cdot 05631)\cos 3\theta - (9\cdot 93475)\frac{p_3}{m'} - (8\cdot 91803)\frac{q^3}{m'} \\
&+(9\cdot 08947)\left\{\frac{p_3}{m'}\cos\theta - \frac{q_3}{m'}\sin\theta\right\} - (9\cdot 14306)\left\{\frac{p_3}{m'}\sin\theta + \frac{q_3}{m'}\cos\theta\right\} \\
&-(8\cdot 65341)\left\{\frac{p_3}{m'}\cos 2\theta - \frac{q_3}{m'}\sin 2\theta\right\} - (8\cdot 87892)\left\{\frac{p_3}{m'}\sin 2\theta + \frac{q_3}{m'}\cos 2\theta\right\}
\end{aligned}$$

$$\begin{aligned}
0 = &\ \ (1\cdot 79213)\sin\theta - (2\cdot 49403)\cos\theta - (2\cdot 55700)\sin 2\theta - (2\cdot 56972)\cos 2\theta \\
&-(2\cdot 20337)\sin 3\theta - (2\cdot 25714)\cos 3\theta + (9\cdot 83632)\frac{p_3}{m'} + (0\cdot 31156)\frac{q_3}{m'} \\
&-(9\cdot 60395)\left\{\frac{p_3}{m'}\cos\theta - \frac{q_3}{m'}\sin\theta\right\} + (9\cdot 47665)\left\{\frac{p_3}{m'}\sin\theta + \frac{q_3}{m'}\cos\theta\right\} \\
&+(9\cdot 23220)\left\{\frac{p_3}{m'}\cos 2\theta - \frac{q_3}{m'}\sin 2\theta\right\} + (9\cdot 21679)\left\{\frac{p_3}{m'}\sin 2\theta + \frac{q_3}{m'}\cos 2\theta\right\}
\end{aligned}$$

where the numbers enclosed within parentheses denote the logarithms of the corresponding coefficients.

30. These equations may be rapidly solved by approximation. The coefficients of $\frac{p_3}{m'}$ and $\frac{q_3}{m'}$ in the first equation being small, we may find from it an approximate value of θ, the substitution of which in the second and third equations will give approximate values of $\frac{p_3}{m'}$ and $\frac{q_3}{m'}$. By means of these a more accurate value of θ may be found from the first equation, and the process being repeated will enable us to satisfy all the equations as nearly as we please.

Thus we find $\theta = -51^\circ\, 30'$, $\quad \frac{p_3}{m'} = 271''\!\cdot 57$, $\quad \frac{q_3}{m'} = -207''\!\cdot 24$.

Now ε is known and $= 217^\circ\, 55'$ $\therefore$ $\varepsilon' = 269^\circ\, 25'$ the mean longitude of the disturbing planet at the epoch $1810\cdot 328$. The sidereal motion in 36 synodic periods of Uranus $= 55^\circ\, 12'$, Precession $= 30'$, $\therefore$ Mean Longitude at the time $1846\cdot 762$, or October 6, 1846, $= 325^\circ\, 7'$.

Also, the analytical expressions for $\frac{p_3}{m'}$ and $\frac{q_3}{m'}$ are

$$\frac{p_3}{m'} = 48''\!\cdot 55 \sin (3\,\theta - \beta) - 93''\!\cdot 01\, e' \sin (3\,\theta - \beta')$$

$$\frac{q_3}{m'} = 48\cdot 55 \cos (3\,\theta - \beta) - 93\cdot 01\, e' \cos (3\,\theta - \beta')$$

where $\varepsilon - \varpi' = \beta'$. Equating these to the values given above, we find $e' = 3\cdot 2206$, $\beta' = 262^\circ\, 28'$, and $\therefore \varpi' = 315^\circ\, 27'$. Hence long. of Perihelion in $1846 = 315^\circ\, 57'$.

Lastly, substituting the values just obtained in equation (1), we find $m' = 0\cdot 82816$.

31. Hence the values of the mass and elements of the orbit of the disturbing planet, resulting from the first hypothesis as to the mean distance, are the following:

$$\frac{a}{a'} = 0\cdot 5$$

	°	′
Mean Long. of the planet, October 6, 1846,	325	7
Longitude of the Perihelion - - - - -	315	57
Eccentricity of the Orbit - - - - - -	0·16103	
Mass (that of the Sun being 1) - - - -	0·0001656	

These are the results which I communicated to the Astronomer Royal in October, 1845.

32. I next entered upon a similar investigation, founded on the assumption that the mean distance was about $\frac{1}{30}$th part less than before, so that $\frac{a}{a'}$ or $\alpha = \sin 31^\circ = 0\cdot 51\dot{5}$. The method employed was, in principle, exactly the same as that given before; but the numerical calculations were somewhat shortened by a few alterations in the process, which had been suggested by my previous solution.

33. Assuming then that $\alpha = \sin 31°$, the values of the quantities b, $\alpha \frac{db}{d\alpha}$, $\alpha^2 \frac{d^2b}{d\alpha^2}$ will be

$$\log. b_0 = 0 \cdot 33385; \quad \log. \alpha \frac{db_0}{d\alpha} = 9 \cdot 57333; \quad \log. \alpha^2 \frac{d^2b_0}{d\alpha^2} = 9 \cdot 82911$$

$$\log. b_1 = 9 \cdot 76106; \quad \log. \alpha \frac{db_1}{d\alpha} = 9 \cdot 86149; \quad \log. \alpha^2 \frac{d^2b_1}{d\alpha^2} = 9 \cdot 76573$$

$$\log. b_2 = 9 \cdot 35361; \quad \log. \alpha \frac{db_2}{d\alpha} = 9 \cdot 71359; \quad \log. \alpha^2 \frac{d^2b_2}{d\alpha^2} = 9 \cdot 92466$$

$$\log. b_3 = 8 \cdot 98918; \quad \log. \alpha \frac{db_3}{d\alpha} = 9 \cdot 50854; \quad \log. \alpha^2 \frac{d^2b_3}{d\alpha^2} = 9 \cdot 91563$$

Hence, by means of the formulæ given before, the principal inequalities of the mean longitude of URANUS, produced by the action of a planet whose mass is $\frac{m'}{5000}$, that of the SUN being unity, and the eccentricity of whose orbit is $\frac{e'}{20}$, may be found to be the following:

$$\begin{array}{lll}
-42''\cdot 33\ m' & \sin & \{nt - n't + \varepsilon - \varepsilon'\} \\
+76\cdot 55\ m' & \sin 2 & \{nt - n't + \varepsilon - \varepsilon'\} \\
+\ 7\cdot 25\ m' & \sin 3 & \{nt - n't + \varepsilon - \varepsilon'\} \\
+\ 2\cdot 34\ m' & \sin & \{n't + \varepsilon' - \varpi\} \\
-\ 4\cdot 74\ m'e' & \sin & \{n't + \varepsilon' - \varpi'\} \\
+41\cdot 72\ m' & \sin & \{nt - 2n't + \varepsilon - 2\varepsilon' + \varpi\} \\
-16\cdot 47\ m'e' & \sin & \{nt - 2n't + \varepsilon - 2\varepsilon' + \varpi'\} \\
+33\cdot 93\ m' & \sin & \{2nt - 3n't + 2\varepsilon - 3\varepsilon' + \varpi\} \\
-63\cdot 41\ m'e' & \sin & \{2nt - 3n't + 2\varepsilon - 3\varepsilon' + \varpi'\}
\end{array}$$

To these we may add the following, which are of two dimensions in terms of the eccentricities:

$$\begin{array}{lll}
+0''\cdot 40\ m' & \sin 3 & \{nt - n't + \varepsilon - \varepsilon'\} \\
-0\cdot 74\ m'e' & \sin & \{3(nt - n't + \varepsilon - \varepsilon') - \varpi + \varpi'\}
\end{array}$$

34. Now, on our present assumption, $n = 13° \ 0' \cdot 6$, $n' = 4° \ 48' \cdot 5$, $n - n' = 8° \ 12' \cdot 1$, $n - 2n' = 3° \ 23' \cdot 6$, $2n - 3n' = 11° \ 35' \cdot 7$.

Hence the equations of condition given by the modern observations will be of the form

$$\begin{aligned}
c'' = \ & \delta\varepsilon + \delta x_1 \cos\{13° \ 0' \cdot 5\} t + \delta x_2 \cos\{26° \ 1' \cdot 0\} t \\
& + t\,\delta n + \delta y_1 \sin\{13 \ 0 \cdot 5\} t + \delta y_2 \sin\{26 \ 1 \cdot 0\} t \\
& + h_1 \cos\{8 \ 12 \cdot 1\} t + h_2 \cos\{16 \ 24 \cdot 2\} t + h_3 \cos\{24° \ 36' \cdot 3\} t \\
& + k_1 \sin\{8 \ 12 \cdot 1\} t + k_2 \sin\{16 \ 24 \cdot 2\} t + k_3 \sin\{24 \ 36 \cdot 3\} t \\
& + p_1 \cos\{4 \ 48 \cdot 5\} t + p_2 \cos\{3 \ 23 \cdot 6\} t + p_3 \cos\{11 \ 35 \cdot 7\} t \\
& + q_1 \sin\{4 \ 48 \cdot 5\} t + q_2 \sin\{3 \ 23 \cdot 6\} t + q_3 \sin\{11 \ 35 \cdot 7\} t
\end{aligned}$$

35. Treating these equations of condition in the same manner as before, the equations in the first group, derived from them, are found to be the following:

$$
\begin{array}{llrrr}
(\epsilon) & 151''{\cdot}48 = & 21{\cdot}0000\,\delta\epsilon + & 6{\cdot}0670\ \delta x_1 - & 4{\cdot}4358\,\delta x_2 \\
& & +13{\cdot}9515\,h_1 + & 0{\cdot}9471\ \ h_2 - & 4{\cdot}5965\ h_3 \\
& & +18{\cdot}3916\,p_1 + & 19{\cdot}6752\ \ p_2 + & 8{\cdot}4184\ p_3 \\
(x) & 216{\cdot}48 = & 6{\cdot}0670\,\delta\epsilon + & 8{\cdot}2821\ \delta x_1 + & 4{\cdot}1762\,\delta x_2 \\
& & +\ 7{\cdot}3540\,h_1 + & 8{\cdot}3027\ \ h_2 + & 5{\cdot}0961\ h_3 \\
& & +\ 6{\cdot}5793\,p_1 + & 6{\cdot}3319\ \ p_2 + & 8{\cdot}0850\ p_3 \\
(h_1) & 207{\cdot}58 = & 13{\cdot}9515\,\delta\epsilon + & 7{\cdot}3540\ \delta x_1 - & 0{\cdot}4177\,\delta x_2 \\
& & +10{\cdot}9735\,h_1 + & 4{\cdot}6775\ \ h_2 - & 0{\cdot}0005\ h_3 \\
& & +12{\cdot}8697\,p_1 + & 13{\cdot}4050\ \ p_2 + & 8{\cdot}4781\ p_3 \\
(h_2) & 245{\cdot}17 = & 0{\cdot}9471\,\delta\epsilon + & 8{\cdot}3027\ \delta x_1 + & 7{\cdot}2362\,\delta x_2 \\
& & +\ 4{\cdot}6775\,h_1 + & 10{\cdot}0259\ \ h_2 + & 8{\cdot}3220\ h_3 \\
& & +\ 2{\cdot}3661\,p_1 + & 1{\cdot}6727\ \ p_2 + & 7{\cdot}3073\ p_3 \\
(h_3) & 103{\cdot}48 = & -\ 4{\cdot}5965\,\delta\epsilon + & 5{\cdot}0961\ \delta x_1 + & 10{\cdot}5558\,\delta x_2 \\
& & -\ 0{\cdot}0005\,h_1 + & 8{\cdot}3220\ \ h_2 + & 10{\cdot}9749\ h_3 \\
& & -\ 2{\cdot}8935\,p_1 - & 3{\cdot}7316\ \ p_2 + & 3{\cdot}5852\ p_3
\end{array}
$$

36. Similarly the equations in the second group, are

$$
\begin{array}{llrrr}
(n) & -171{\cdot}27 = & 77{\cdot}0000\,\delta n + & 9{\cdot}3938\ \delta y_1 - & 1{\cdot}2183\,\delta y_2 \\
& & +\ 8{\cdot}7355\,k_1 + & 7{\cdot}6213\ \ k_2 - & 0{\cdot}0590\ k_3 \\
& & +\ 5{\cdot}9764\,q_1 + & 4{\cdot}3875\ \ q_2 + & 9{\cdot}6152\ q_3 \\
(y) & -166{\cdot}33 = & 93{\cdot}9380\,\delta n + & 12{\cdot}7179\ \delta y_1 + & 1{\cdot}8907\,\delta y_2 \\
& & +11{\cdot}0393\,k_1 + & 11{\cdot}3717\ \ k_2 + & 3{\cdot}3196\ k_3 \\
& & +\ 7{\cdot}3747\,q_1 + & 5{\cdot}3825\ \ q_2 + & 12{\cdot}6816\ q_3 \\
(k_1) & -181{\cdot}31 = & 87{\cdot}3550\,\delta n + & 11{\cdot}0393\ \delta y_1 - & 0{\cdot}3758\,\delta y_2 \\
& & +10{\cdot}0264\,k_1 + & 9{\cdot}2740\ \ k_2 + & 0{\cdot}9476\ k_3 \\
& & +\ 6{\cdot}8054\,q_1 + & 4{\cdot}9866\ \ q_2 + & 11{\cdot}1971\ q_3 \\
(k_2) & -\ 99{\cdot}51 = & 76{\cdot}2130\,\delta n + & 11{\cdot}3717\ \delta y_1 + & 4{\cdot}4810\,\delta y_2 \\
& & +\ 9{\cdot}2740\,k_1 + & 10{\cdot}9740\ \ k_2 + & 5{\cdot}6294\ k_3 \\
& & +\ 6{\cdot}0523\,q_1 + & 4{\cdot}3916\ \ q_2 + & 11{\cdot}0843\ q_3 \\
(k_3) & 113{\cdot}14 = & -\ 0{\cdot}5900\,\delta n + & 3{\cdot}3196\ \delta y_1 + & 10{\cdot}2112\,\delta y_2 \\
& & +\ 0{\cdot}9476\,k_1 + & 5{\cdot}6294\ \ k_2 + & 10{\cdot}0251\ k_3 \\
& & +\ 0{\cdot}1746\,q_1 + & 0{\cdot}0454\ \ q_2 + & 2{\cdot}4791\ q_3
\end{array}
$$

37. The equations (p_2), (p_3) of the first group, and (q_2), (q_3) of the second were not formed, as our previous solution shewed that when $\delta\epsilon$, δn, δx_1, and δy_1, were eliminated, the co-efficients of the remaining unknown quantities in these equations would be extremely small. It will be preferable to combine the equations (h_1), (h_2), (h_3), and (k_1), (k_2), (k_3) before, instead of after, the elimination of $\delta\epsilon$, δn, δx_1, and δy_1, from them.

If then we change the sign of the third equation in each group, and add it to the fourth and fifth, we obtain

$$\begin{aligned} 111''{\cdot}07 = &-17{\cdot}6009\ \delta\epsilon + 6{\cdot}0448\ \delta x_1 + 18{\cdot}2097\ \delta x_2 \\ &- 6{\cdot}2965\ h_1 + 13{\cdot}6704\ h_2 + 19{\cdot}2974\ h_3 \\ &-13{\cdot}3971\ p_1 - 15{\cdot}4639\ p_2 + 2{\cdot}4144\ p_3 \end{aligned}$$

$$\begin{aligned} 194{\cdot}94 = &-11{\cdot}7320\ \delta n + 3{\cdot}6520\ \delta y_1 + 15{\cdot}0680\ \delta y_2 \\ &+ 0{\cdot}1951\ k_1 + 7{\cdot}3294\ k_2 + 14{\cdot}7069\ k_3 \\ &- 0{\cdot}5785\ q_1 - 0{\cdot}5496\ q_2 + 2{\cdot}3663\ q_3 \end{aligned}$$

38. By means of (ϵ) and (n) of Articles 35 and 36, eliminate $\delta\epsilon$ and δn from (x) and (y), and also from the equations just found, and we have

$$\begin{aligned} (x)\ 202''{\cdot}72 = &\ 6{\cdot}5294\ \delta x_1 + 5{\cdot}4577\ \delta x_2 + 3{\cdot}3234\ h_1 + 8{\cdot}0291\ h_2 \\ &+ 6{\cdot}4240\ h_3 + 1{\cdot}2659\ p_1 + 0{\cdot}6477\ p_2 + 5{\cdot}6529\ p_3 \end{aligned}$$

$$\begin{aligned} (y)\ 42{\cdot}61 = &\ 1{\cdot}2578\ \delta y_1 + 3{\cdot}3771\ \delta y_2 + 0{\cdot}3822\ k_1 + 2{\cdot}0739\ k_2 \\ &+ 3{\cdot}3916\ k_3 + 0{\cdot}0836\ q_1 + 0{\cdot}0298\ q_2 + 0{\cdot}9513\ q_3 \end{aligned}$$

$$\begin{aligned} 268{\cdot}02 = &\ 11{\cdot}1297\ \delta x_1 + 14{\cdot}4919\ \delta x_2 + 5{\cdot}3967\ h_1 + 14{\cdot}4642\ h_2 \\ &+ 15{\cdot}4449\ h_3 + 2{\cdot}0175\ p_1 + 1{\cdot}0266\ p_2 + 9{\cdot}4702\ p_3 \end{aligned}$$

$$\begin{aligned} 168{\cdot}85 = &\ 5{\cdot}0833\ \delta y_1 + 14{\cdot}8824\ \delta y_2 + 1{\cdot}5261\ k_1 + 8{\cdot}4906\ k_2 \\ &+ 14{\cdot}6979\ k_3 + 0{\cdot}3320\ q_1 + 0{\cdot}1189\ q_2 + 3{\cdot}8313\ q_3 \end{aligned}$$

39. Substituting for δx_2, δy_2 their values in terms of δx_1, δy_1, we find

$$\begin{aligned} 6{\cdot}5294\ \delta x_1 + 5{\cdot}4577\ \delta x_2 &= 6{\cdot}5700\ \delta x_1 + 0{\cdot}0490\ \delta y_1 \\ 1{\cdot}2578\ \delta y_1 + 3{\cdot}3771\ \delta y_2 &= -0{\cdot}0303\ \delta x_1 + 1{\cdot}2829\ \delta y_1 \\ 11{\cdot}1297\ \delta x_1 + 14{\cdot}4919\ \delta x_2 &= 11{\cdot}2378\ \delta x_1 + 0{\cdot}1300\ \delta y_1 \\ 5{\cdot}0833\ \delta y_1 + 14{\cdot}8824\ \delta y_2 &= -0{\cdot}1335\ \delta x_1 + 5{\cdot}1943\ \delta y_1 \end{aligned}$$

Hence, if we add to the two latter equations

$$-1{\cdot}7106\ (x) - 0{\cdot}03607\ (y)$$

and $$0{\cdot}00165\ (x) - 4{\cdot}0487\ (y) \text{ respectively,}$$

δx_1 and δy_1 will be eliminated, and we shall obtain the following equations:

$$\begin{aligned} (1)\quad 80''{\cdot}28 = &\ 0{\cdot}2883\ h_1 - 0{\cdot}7295\ h_2 - 4{\cdot}4559\ h_3 \\ &+ 0{\cdot}0138\ k_1 + 0{\cdot}0748\ k_2 + 0{\cdot}1223\ k_3 \\ &+ 0{\cdot}1479\ p_1 + 0{\cdot}0813\ p_2 + 0{\cdot}1997\ p_3 \\ &+ 0{\cdot}0030\ q_1 + 0{\cdot}0011\ q_2 + 0{\cdot}0343\ q_3 \end{aligned}$$

$$\begin{aligned} (2)\quad 3{\cdot}34 = &-0{\cdot}0055\ h_1 - 0{\cdot}0132\ h_2 - 0{\cdot}0106\ h_3 \\ &+ 0{\cdot}0212\ k_1 - 0{\cdot}0939\ k_2 - 0{\cdot}9662\ k_3 \\ &- 0{\cdot}0021\ p_1 - 0{\cdot}0011\ p_2 - 0{\cdot}0093\ p_3 \\ &+ 0{\cdot}0066\ q_1 + 0{\cdot}0017\ q_2 + 0{\cdot}0203\ q_3 \end{aligned}$$

40. Again, the equations of condition given by the Ancient Observations are

$$\begin{aligned}
62''\!\cdot\!6 = {} & \delta\varepsilon - 0\cdot8776\ \delta x_1 + 0\cdot5402\ \delta x_2 + 0\cdot7923\ h_1 + 0\cdot2554\ h_2 \\
& - 39\cdot31\ \delta n - 0\cdot4795\ \delta y_1 + 0\cdot8415\ \delta y_2 + 0\cdot6101\ k_1 + 0\cdot9668\ k_2 \\
& - 0\cdot3875\ h_3 - 0\cdot9877\ p_1 - 0\cdot6870\ p_2 - 0\cdot1009\ p_3 \\
& + 0\cdot9219\ k_3 + 0\cdot1566\ q_1 - 0\cdot7267\ q_2 - 0\cdot9949\ q_3
\end{aligned}$$

$$\begin{aligned}
84\cdot5 = {} & \delta\varepsilon + 0\cdot4975\ \delta x_1 - 0\cdot5050\ \delta x_2 - 0\cdot0887\ h_1 - 0\cdot9843\ h_2 \\
& - 32\cdot30\ \delta n - 0\cdot8675\ \delta y_1 - 0\cdot8631\ \delta y_2 + 0\cdot9961\ k_1 - 0\cdot1767\ k_2 \\
& + 0\cdot2634\ h_3 - 0\cdot9085\ p_1 - 0\cdot3355\ p_2 + 0\cdot9681\ p_3 \\
& - 0\cdot9647\ k_3 - 0\cdot4178\ q_1 - 0\cdot9420\ q_2 - 0\cdot2506\ q_3
\end{aligned}$$

$$\begin{aligned}
67\cdot2 = {} & \delta\varepsilon + 0\cdot6732\ \delta x_1 - 0\cdot0935\ \delta x_2 - 0\cdot2243\ h_1 - 0\cdot8994\ h_2 \\
& - 31\cdot34\ \delta n - 0\cdot7394\ \delta y_1 - 0\cdot9956\ \delta y_2 + 0\cdot9745\ k_1 - 0\cdot4371\ k_2 \\
& + 0\cdot6277\ h_3 - 0\cdot8720\ p_1 - 0\cdot2815\ p_2 + 0\cdot9982\ p_3 \\
& - 0\cdot7785\ k_3 - 0\cdot4895\ q_1 - 0\cdot9596\ q_2 - 0\cdot0591\ q_3
\end{aligned}$$

$$\begin{aligned}
-51\cdot8 = {} & \delta\varepsilon - 0\cdot2616\ \delta x_1 - 0\cdot8631\ \delta x_2 - 0\cdot9436\ h_1 + 0\cdot7809\ h_2 \\
& - 19\cdot59\ \delta n + 0\cdot9652\ \delta y_1 - 0\cdot5050\ \delta y_2 - 0\cdot3310\ k_1 + 0\cdot6247\ k_2 \\
& - 0\cdot5301\ h_3 - 0\cdot0731\ p_1 + 0\cdot3991\ p_2 - 0\cdot6801\ p_3 \\
& - 0\cdot8179\ k_3 - 0\cdot9973\ q_1 - 0\cdot9169\ q_2 + 0\cdot7331\ q_3
\end{aligned}$$

$$\begin{aligned}
-43\cdot2 = {} & \delta\varepsilon - 0\cdot4741\ \delta x_1 - 0\cdot5505\ \delta x_2 - 0\cdot8861\ h_1 + 0\cdot5704\ h_2 \\
& - 18\cdot58\ \delta n + 0\cdot8805\ \delta y_1 - 0\cdot8348\ \delta y_2 - 0\cdot4634\ k_1 + 0\cdot8213\ k_2 \\
& - 0\cdot1248\ h_3 + 0\cdot0115\ p_1 + 0\cdot4532\ p_2 - 0\cdot8147\ p_3 \\
& - 0\cdot9922\ k_3 - 0\cdot9999\ q_1 - 0\cdot8914\ q_2 + 0\cdot5798\ q_3
\end{aligned}$$

$$\begin{aligned}
-50\cdot1 = {} & \delta\varepsilon - 0\cdot6430\ \delta x_1 - 0\cdot1731\ \delta x_2 - 0\cdot8191\ h_1 + 0\cdot3420\ h_2 \\
& - 17\cdot68\ \delta n + 0\cdot7659\ \delta y_1 - 0\cdot9849\ \delta y_2 - 0\cdot5736\ k_1 + 0\cdot9397\ k_2 \\
& + 0\cdot2588\ h_3 + 0\cdot0871\ p_1 + 0\cdot5001\ p_2 - 0\cdot9063\ p_3 \\
& - 0\cdot9659\ k_3 - 0\cdot9962\ q_1 - 0\cdot8660\ q_2 + 0\cdot4225\ q_3
\end{aligned}$$

$$\begin{aligned}
-37\cdot8 = {} & \delta\varepsilon - 0\cdot9492\ \delta x_1 + 0\cdot8021\ \delta x_2 - 0\cdot5743\ h_1 - 0\cdot3404\ h_2 \\
& - 15\cdot25\ \delta n + 0\cdot3145\ \delta y_1 - 0\cdot5972\ \delta y_2 - 0\cdot8186\ k_1 + 0\cdot9403\ k_2 \\
& + 0\cdot9652\ h_3 + 0\cdot2872\ p_1 + 0\cdot6192\ p_2 - 0\cdot9984\ p_3 \\
& - 0\cdot2613\ k_3 - 0\cdot9579\ q_1 - 0\cdot7852\ q_2 - 0\cdot0560\ q_3
\end{aligned}$$

$$\begin{aligned}
-20\cdot5 = {} & \delta\varepsilon - 0\cdot9985\ \delta x_1 + 0\cdot9942\ \delta x_2 - 0\cdot3671\ h_1 - 0\cdot7304\ h_2 \\
& - 13\cdot60\ \delta n - 0\cdot0538\ \delta y_1 + 0\cdot1074\ \delta y_2 - 0\cdot9302\ k_1 + 0\cdot6830\ k_2 \\
& + 0\cdot9035\ h_3 + 0\cdot4164\ p_1 + 0\cdot6928\ p_2 - 0\cdot9251\ p_3 \\
& + 0\cdot4286\ k_3 - 0\cdot9092\ q_1 - 0\cdot7212\ q_2 - 0\cdot3796\ q_3
\end{aligned}$$

$$\begin{aligned}
-2\cdot4 = {} & \delta\varepsilon - 0\cdot9633\ \delta x_1 + 0\cdot8560\ \delta x_2 - 0\cdot2363\ h_1 - 0\cdot8883\ h_2 \\
& - 12\cdot64\ \delta n - 0\cdot2684\ \delta y_1 + 0\cdot5170\ \delta y_2 - 0\cdot9717\ k_1 + 0\cdot4593\ k_2 \\
& + 0\cdot6562\ h_3 + 0\cdot4882\ p_1 + 0\cdot7327\ p_2 - 0\cdot8345\ p_3 \\
& + 0\cdot7546\ k_3 - 0\cdot8727\ q_1 - 0\cdot6806\ q_2 - 0\cdot5511\ q_3
\end{aligned}$$

41. The equation for finding p_3 may be formed, as before, by multiplying the above equations taken in order by

$$-0 \cdot 8, \ -0 \cdot 6, \ +1 \cdot 0, \ +1 \cdot 0, \ +0 \cdot 9, \ +0 \cdot 6, \ +0 \cdot 4, \ +0 \cdot 3$$

beginning with the second; and the equation for q_3 by multiplying the same equations by

$$1 \cdot 0, \ \ 1 \cdot 0, \ \ 0 \cdot 5, \ \ 0 \cdot 4, \ \ 0 \cdot 3, \ \ 0 \cdot 2, \ \ 0 \cdot 1, \ \ 0 \cdot 1.$$

Thus we obtain

$$\begin{aligned} -279''\cdot 64 = {} & 2 \cdot 80\, \delta\varepsilon - 3 \cdot 3742\, \delta x_1 + 0 \cdot 0265\, \delta x_2 - 2 \cdot 9237\, h_1 + 2 \cdot 2232\, h_2 \\ & -27 \cdot 82\, \delta n + 3 \cdot 7593\, \delta y_1 - 1 \cdot 0986\, \delta y_2 - 3 \cdot 8471\, k_1 + 3 \cdot 6706\, k_2 \\ & + 0 \cdot 1281\, h_3 + 1 \cdot 7522\, p_1 + 2 \cdot 6081\, p_2 - 4 \cdot 9033\, p_3 \\ & - 1 \cdot 2295\, k_3 - 3 \cdot 4661\, q_1 - 2 \cdot 2221\, q_2 + 1 \cdot 5785\, q_3 \end{aligned}$$

$$\begin{aligned} 83 \cdot 56 = {} & 3 \cdot 60\, \delta\varepsilon + 0 \cdot 2714\, \delta x_1 - 0 \cdot 9567\, \delta x_2 - 1 \cdot 5602\, h_1 - 1 \cdot 3924\, h_2 \\ & -91 \cdot 84\, \delta n - 0 \cdot 5116\, \delta y_1 - 2 \cdot 7976\, \delta y_2 + 1 \cdot 0937\, k_1 + 0 \cdot 6112\, k_2 \\ & + 1 \cdot 0027\, h_3 - 1 \cdot 6385\, p_1 + 0 \cdot 1802\, p_2 + 0 \cdot 6529\, p_3 \\ & - 2 \cdot 7879\, k_3 - 2 \cdot 4746\, q_1 - 3 \cdot 2736\, q_2 + 0 \cdot 3113\, q_3 \end{aligned}$$

42. Eliminate $\delta\varepsilon$ and δn by means of (ε) and (n) of Articles 35 and 36, and these equations become

$$\begin{aligned} -361''\cdot 72 = {} & - 4 \cdot 1831\, \delta x_1 + 0 \cdot 6179\, \delta x_2 - 4 \cdot 7839\, h_1 + 2 \cdot 0969\, h_2 \\ & + 7 \cdot 1533\, \delta y_1 - 1 \cdot 5388\, \delta y_2 - 0 \cdot 6909\, k_1 + 6 \cdot 4242\, k_2 \\ & + 0 \cdot 7410\, h_3 - 0 \cdot 7000\, p_1 - 0 \cdot 0153\, p_2 - 6 \cdot 0258\, p_3 \\ & - 1 \cdot 2508\, k_3 - 1 \cdot 3068\, q_1 - 0 \cdot 6369\, q_2 + 5 \cdot 0525\, q_3 \end{aligned}$$

$$\begin{aligned} -146 \cdot 69 = {} & - 0 \cdot 7686\, \delta x_1 - 0 \cdot 1963\, \delta x_2 - 3 \cdot 9519\, h_1 - 1 \cdot 5548\, h_2 \\ & + 10 \cdot 6926\, \delta y_1 - 4 \cdot 2508\, \delta y_2 + 11 \cdot 5128\, k_1 + 9 \cdot 7013\, k_2 \\ & + 1 \cdot 7907\, h_3 - 4 \cdot 7913\, p_1 - 3 \cdot 1927\, p_2 - 0 \cdot 7902\, p_3 \\ & - 2 \cdot 8583\, k_3 + 4 \cdot 6536\, q_1 + 1 \cdot 9595\, q_2 + 11 \cdot 7796\, q_3 \end{aligned}$$

43. Substituting for δx_2, δy_2, their values in terms of δx_1, δy_1, we find

$$-4 \cdot 1831\, \delta x_1 + 7 \cdot 1533\, \delta y_1 + 0 \cdot 6179\, \delta x_2 - 1 \cdot 5388\, \delta y_2 = -4 \cdot 1647\, \delta x_1 + 7 \cdot 1473\, \delta y_1$$

$$-0 \cdot 7686\, \delta x_1 + 10 \cdot 6926\, \delta y_1 - 0 \cdot 1963\, \delta x_2 - 4 \cdot 2508\, \delta y_2 = -0 \cdot 7319\, \delta x_1 + 10 \cdot 6591\, \delta y_1$$

Hence, if to the equations just found, we add

$$+0 \cdot 60808\ (x) - 5 \cdot 5942\ (y)$$

and $$+0 \cdot 07306\ (x) - 8 \cdot 3110\ (y) \text{ respectively,}$$

δx_1 and δy_1 will be eliminated, and we shall obtain the following equations:

$$\begin{aligned} (3) \qquad -476''\cdot 84 = {} & -2{\cdot}7630\, h_1 + 6{\cdot}9793\, h_2 + 4{\cdot}6473\, h_3 \\ & -2{\cdot}8290\, k_1 - 5{\cdot}1777\, k_2 - 20{\cdot}2242\, k_3 \\ & +0{\cdot}0698\, p_1 + 0{\cdot}3785\, p_2 - 2{\cdot}5884\, p_3 \\ & -1{\cdot}7748\, q_1 - 0{\cdot}8036\, q_2 - 0{\cdot}2693\, q_3 \end{aligned}$$

$$(4) \qquad -486''{\cdot}03 = -3{\cdot}7091\,h_1 - 0{\cdot}9682\,h_2 + 2{\cdot}2600\,h_3$$
$$+8{\cdot}3364\,k_1 - 7{\cdot}5348\,k_2 - 31{\cdot}0457\,k_3$$
$$-4{\cdot}6988\,p_1 - 3{\cdot}1454\,p_2 - 0{\cdot}3772\,p_3$$
$$+3{\cdot}9584\,q_1 + 1{\cdot}7118\,q_2 + 3{\cdot}8734\,q_3$$

44. Eliminate the left hand members from equations (2), (3), and (4) of Articles 39 and 43, by means of equation (1), and we have

$$0 = 0\,{\cdot}4200\,h_1 - 0\,{\cdot}4114\,h_2 - 4{\cdot}2014\,h_3 + 0\,{\cdot}1980 p_1 + 0\,{\cdot}1069\,p_2 + 0\,{\cdot}4236\,p_3$$
$$-0\,{\cdot}4964\,k_1 + 2\,{\cdot}3306\,k_2 + 23{\cdot}3213\,k_3 - 0\,{\cdot}1567\,q_1 - 0\,{\cdot}0409\,q_2 - 0\,{\cdot}4531\,q_3$$

$$0 = -1\,{\cdot}0507\,h_1 + 2\,{\cdot}6465\,h_2 - 21{\cdot}8182\,h_3 + 0\,{\cdot}9482\,p_1 + 0\,{\cdot}8614\,p_2 - 1\,{\cdot}4023\,p_3$$
$$-2\,{\cdot}7471\,k_1 - 4\,{\cdot}7334\,k_2 - 19{\cdot}4976\,k_3 - 1\,{\cdot}7569\,q_1 - 0\,{\cdot}7972\,q_2 - 0\,{\cdot}0655 q_3$$

$$0 = -1\,{\cdot}9638\,h_1 - 5\,{\cdot}3845 h_2 - 24{\cdot}7155\,h_3 - 3\,{\cdot}8034\,p_1 - 2\,{\cdot}6532 p_2 + 0\,{\cdot}8317\,p_3$$
$$+8\,{\cdot}4199\,k_1 - 7\,{\cdot}0819\,k_2 - 30{\cdot}3051\,k_3 + 3\,{\cdot}9767\,q_1 + 1\,{\cdot}7183\,q_2 + 4\,{\cdot}0811\,q_3$$

45. If, as before, we put $\varepsilon - \varepsilon' = \theta$ and $\varepsilon - \varpi = \beta$, it may be seen that

$$\frac{h^1}{m'} = -42''{\cdot}33 \sin\theta, \qquad \frac{h_2}{m'} = 76''{\cdot}55 \sin 2\theta$$

$$\frac{k_1}{m'} = -42\,{\cdot}33 \cos\theta, \qquad \frac{k_2}{m'} = 76\,{\cdot}55 \cos 2\theta$$

$$\frac{h_3}{m'} = 7''{\cdot}25 \sin 3\theta + 0\,{\cdot}007460\,\frac{p_3}{m'} + 0\,{\cdot}008974\,\frac{q_3}{m'}$$

$$\frac{h_3}{m'} = 7\,{\cdot}25 \cos 3\theta - 0\,{\cdot}008974\,\frac{p_3}{m'} + 0\,{\cdot}007460\,\frac{q_3}{m'}$$

$$\frac{p_1}{m'} = 0\,{\cdot}20 \sin(\theta-\beta) - 0\,{\cdot}074738 \left\{\frac{p_3}{m'}\cos 2\theta - \frac{q_3}{m'}\sin 2\theta\right\}$$

$$\frac{q_1}{m'} = -\,0\,{\cdot}20 \cos(\theta-\beta) + 0\,{\cdot}074738 \left\{\frac{p_3}{m'}\sin 2\theta + \frac{q_3}{m'}\cos 2\theta\right\}$$

$$\frac{p_2}{m'} = 32\,{\cdot}91 \sin(2\theta-\beta) + 0\,{\cdot}259765 \left\{\frac{p_3}{m'}\cos\theta - \frac{q_3}{m'}\sin\theta\right\}$$

$$\frac{q_2}{m'} = 32\,{\cdot}91 \cos(2\theta-\beta) + 0\,{\cdot}259765 \left\{\frac{p_3}{m'}\sin\theta + \frac{q_3}{m'}\cos\theta\right\}$$

46. Substituting these expressions in the above equations, and putting for β its value $50^\circ\ 15'\ {\cdot}8$, we obtain

$$0 = -(1\,{\cdot}24872)\sin\theta + (1\,{\cdot}32231)\cos\theta - (1\,{\cdot}48110)\sin 2\theta + (2\,{\cdot}24265)\cos 2\theta$$

$$-(1\,{\cdot}48373)\sin 3\theta + (2\,{\cdot}22809)\cos 3\theta + (9\,{\cdot}26254)\,\frac{p_3}{m'} - (9\,{\cdot}50079)\,\frac{q_3}{m'}$$

$$+(8\,{\cdot}44376)\left\{\frac{p_3}{m'}\cos\theta - \frac{q_3}{m'}\sin\theta\right\} - (8\,{\cdot}02630)\left\{\frac{p_3}{m'}\sin\theta + \frac{q_3}{m'}\cos\theta\right\}$$

$$-(8\,{\cdot}17031)\left\{\frac{p_3}{m'}\cos 2\theta - \frac{q_3}{m'}\sin 2\theta\right\} - (8\,{\cdot}06861)\left\{\frac{p_3}{m'}\sin 2\theta + \frac{q_3}{m'}\cos 2\theta\right\}$$

$$0 = (1 \cdot 65190) \sin\theta + (2 \cdot 06584) \cos\theta + (2 \cdot 30220) \sin 2\theta - (2 \cdot 60306) \cos 2\theta$$
$$- (2 \cdot 19916) \sin 3\theta - (2 \cdot 15032) \cos 3\theta - (0 \cdot 14305) \frac{p_3}{m'} - (9 \cdot 60933) \frac{q_3}{m'}$$
$$+ (9 \cdot 34981) \left\{\frac{p_3}{m'} \cos\theta - \frac{q_3}{m'} \sin\theta\right\} - (9 \cdot 31615) \left\{\frac{p_3}{m'} \sin\theta + \frac{q_3}{m'} \cos\theta\right\}$$
$$- (8 \cdot 85046) \left\{\frac{p_3}{m'} \cos 2\theta - \frac{q_3}{m'} \sin 2\theta\right\} - (9 \cdot 11828) \left\{\frac{p^3}{m'} \sin 2\theta + \frac{q_3}{m'} \cos 2\theta\right\}$$

$$0 = (1 \cdot 91407) \sin\theta - (2 \cdot 55189) \cos\theta - (2 \cdot 62790) \sin 2\theta - (2 \cdot 64230) \cos 2\theta$$
$$- (2 \cdot 25331) \sin 3\theta - (2 \cdot 34185) \cos 3\theta + (9 \cdot 96344) \frac{p_3}{m'} + (0 \cdot 56029) \frac{q_3}{m'}$$
$$- (9 \cdot 83835) \left\{\frac{p_3}{m'} \cos\theta - \frac{q_3}{m'} \sin\theta\right\} + (9 \cdot 64968) \left\{\frac{p_3}{m'} \sin\theta + \frac{q_3}{m'} \cos\theta\right\}$$
$$+ (9 \cdot 45371) \left\{\frac{p_3}{m'} \cos 2\theta - \frac{q_3}{m'} \sin 2\theta\right\} + (9 \cdot 47306) \left\{\frac{p_3}{m'} \sin 2\theta + \frac{q_3}{m'} \cos 2\theta\right\}$$

where the numbers enclosed within parentheses denote the logarithms of the corresponding co-efficients, as before.

47. From these equations, we find, by the same method as before

$$\theta = -46^\circ\, 55', \quad \frac{p_3}{m'} = 138''\cdot 92, \quad \frac{q_3}{m'} = -109''\cdot 83$$

Hence, since $\varepsilon = 217^\circ\, 55'$, $\varepsilon' = 264^\circ\, 50'$, the mean longitude of the disturbing planet at the epoch 1810 ·328. The sidereal motion in 36 synodic periods of Uranus $= 57^\circ\, 42'$, Precession $= 30'$. $\therefore$ Mean Longitude at the time 1846 ·762, or October 6, 1846, $= 323^\circ\, 2'$.

Also, the expressions for $\frac{p_3}{m'}$ and $\frac{q_3}{m'}$ are

$$\frac{p_3}{m'} = 33''\cdot 93 \sin(3\theta - \beta) - 63''\cdot 41\, e' \sin(3\theta - \beta')$$

$$\frac{q_3}{m'} = 33 \cdot 93 \cos(3\theta - \beta) - 63 \cdot 41\, e' \cos(3\theta - \beta')$$

where $\varepsilon - \varpi' = \beta'$.

Equating these to the values given above, we find $e' = 2 \cdot 4123$, $\beta' = 279^\circ\, 14'$, and $\therefore \varpi' = 298^\circ\, 41'$. Hence longitude of the perihelion in 1846 $= 299^\circ\, 11'$.

Lastly, substituting the values just obtained in equation (1) of Article 39, we find $m' = 0 \cdot 75017$.

48. Hence the values of the mass and elements of the orbit of the disturbing planet, resulting from the second hypothesis as to the mean distance, are the following:

$$\frac{a}{a'} = 0 \cdot 515$$

	° ′
Mean Longitude of the Planet, October 6, 1846,	323 2
Longitude of the Perihelion - - - - - -	299 11
Eccentricity of the Orbit - - - - - - -	0 ·120615
Mass (that of the Sun being 1) - - - -	0 ·00015003

49. From the values of m', θ, $\frac{p_3}{m'}$ and $\frac{q_3}{m'}$ found above, the values of the quantities h, k, p, and q, corresponding to each hypothesis, are immediately determined. Thus we find

1st Hypothesis.		*2nd Hypothesis.*	
$\frac{a}{a'} = 0 \cdot 5$		$\frac{a}{a'} = 0 \cdot 515$	
$h_1 = \quad 23'' \cdot 98$	$k_1 = - \ 19'' \cdot 07$	$h_1 = \quad 23'' \cdot 19$	$k_1 = - 21'' \cdot 69$
$h_2 = - \ 47 \cdot 58$	$k_2 = - \ 11 \cdot 00$	$h_2 = - \ 57 \cdot 30$	$k_2 = - \ 3 \cdot 83$
$h_3 = - \ 1 \cdot 93$	$k_3 = - \ 7 \cdot 64$	$h_3 = - \ 3 \cdot 40$	$k_3 = - \ 5 \cdot 76$
$p_1 = \quad 9 \cdot 93$	$q_1 = - \ 8 \cdot 31$	$p_1 = \quad 6 \cdot 52$	$q_1 = - \ 7 \cdot 34$
$p_2 = - \ 8 \cdot 54$	$q_2 = - \ 55 \cdot 36$	$p_2 = - \ 11 \cdot 62$	$q_2 = - \ 54 \cdot 39$
$p_3 = \quad 224 \cdot 90$	$q_3 = - \ 171 \cdot 63$	$p_3 = \quad 104 \cdot 21$	$q_3 = - \ 82 \cdot 39$

50. And by substituting these values in the equations (ϵ), (n), (x), and (y), we obtain

1st Hypothesis.		*2nd Hypothesis.*	
$\frac{a}{a'} = 0 \cdot 5$		$\frac{a}{a'} = 0 \cdot 515$	
$\delta\epsilon = - \ 49'' \cdot 77$	$\delta n = - 0'' \cdot 702$	$\delta\epsilon = - 43'' \cdot 23$	$\delta n = - 0'' \cdot 5417$
$\delta x_1 = - 130 \cdot 69$	$\delta y_1 = 222 \cdot 38$	$\delta x_1 = \quad 1 \cdot 77$	$\delta y_1 = 123 \cdot 98$
$\delta x_2 = \quad 1 \cdot 02$	$\delta y_2 = \quad 2 \cdot 83$	$\delta x_2 = \quad 1 \cdot 13$	$\delta y_2 = \quad 0 \cdot 91$

and the corresponding corrections of the elliptic elements will be

$\frac{\delta a}{a} = 0 \cdot 00000999$	$\frac{\delta a}{a} = 0 \cdot 00000771$
$\delta e = 20'' \cdot 83$	$\delta e = 40'' \cdot 31$
$e\delta\varpi = 127 \cdot 27$	$e\delta\varpi = 47 \cdot 10$

It will be seen that the corrections of the eccentricity and longitude of perihelion vary very rapidly with a change in the assumed mean distance.

51. If these quantitities be substituted in the expressions before given, we obtain the following theoretical corrections of the mean longitude, each of these corrections being divided into two parts, of which the first is due to the changes in the elements of the orbit of Uranus, and the second to the action of the disturbing planet.

Hypothesis I.

Ancient Observations.

Year.	″ ″ ″
1712	−288·0 + 365·8 = + 77·8
1715	−283·1 + 357·1 = + 74·0
1750	+ 210·5 − 260·7 = − 50·2
1753	+ 218·1 − 267·0 = − 48·9
1756	+ 214·0 − 260·0 = − 46·0
1764	+ 154·0 − 186·7 = − 32·7
1769	+ 79·6 − 100·7 = − 21·1
1771	+ 27·6 − 41·8 = − 14·2

Modern Observations.

Year.	″ ″ ″
1780	−126·12 + 129·27 = + 3·15
1783	−180·28 + 188·70 = + 8·42
1786	−227·66 + 240·36 = + 12·70
1789	−265·70 + 281·63 = + 15·93
1792	−292·25 + 310·38 = + 18·13
1795	−305·84 + 325·27 = + 19·43
1798	−305·67 + 325·72 = + 20·05
1801	−291·77 + 312·05 = + 20·28
1804	−264·95 + 285·38 = + 20·43
1807	−226·78 + 247·51 = + 20·73
1810	−179·43 + 200·76 = + 21·33
1813	−125·59 + 147·72 = + 22·13
1816	− 68·21 + 91·02 = + 22·81
1819	− 10·40 + 33·18 = + 22·78
1822	+ 44·84 − 23·64 = + 21·20
1825	+ 94·69 − 77·64 = + 17·05
1828	+ 136·73 − 127·48 = + 9·25
1831	+ 168·94 − 172·17 = − 3·23
1834	+ 189·85 − 211·04 = − 21·19
1837	+ 198·51 − 243·59 = − 45·08
1840	+ 194·54 − 269·36 = − 74·82

Hypothesis II.

Ancient Observations.

Year.	″ ″ ″
1712	−133·7 + 211·9 = + 78·2
1715	−117·7 + 191·5 = + 73·8
1750	+ 85·2 − 134·4 = − 49·2
1753	+ 73·8 − 122·2 = − 48·4
1756	+ 59·1 − 105·2 = − 46·1
1764	+ 2·7 − 36·4 = − 33·7
1769	− 43·1 + 20·8 = − 22·3
1771	− 69·9 + 54·7 = − 15·2

Modern Observations.

Year.	″ ″ ″
1780	−133·10 + 135·98 = + 2·88
1783	−149·47 + 157·87 = + 8·40
1786	−160·15 + 172·99 = + 12·84
1789	−164·52 + 180·64 = + 16·12
1792	−162·30 + 180·58 = + 18·28
1795	−153·59 + 173·07 = + 19·48
1798	−138·87 + 158·86 = + 19·99
1801	−118·95 + 139·08 = + 20·13
1804	− 94·96 + 115·21 = + 20·25
1807	− 68·25 + 88·85 = + 20·60
1810	− 40·33 + 61·61 = + 21·28
1813	− 12·72 + 34·91 = + 22·19
1816	+ 13·08 + 9·88 = + 22·96
1819	+ 35·71 − 12·74 = + 22·97
1822	+ 54·04 − 32·68 = + 21·36
1825	+ 67·18 − 50·08 = + 17·10
1828	+ 74·52 − 65·37 = + 9·15
1831	+ 75·74 − 79·21 = − 3·47
1834	+ 70·85 − 92·31 = − 21·46
1837	+ 60·08 − 105·25 = − 45·17
1840	+ 43·98 − 118·38 = − 74·40

52. Comparing these with the corrections of mean longitude derived from observation, we find the remaining differences to be the following:

Ancient Observations.

Year.	Observation—Theory. Hypoth. I.	Hypoth. II.
	″	″
1712	+ 6·7	+ 6·3
1715	− 6·8	− 6·6
1750	− 1·6	− 2·6
1753	+ 5·7	+ 5·2
1756	− 4·1	− 4·0
1764	− 5·1	− 4·1
1769	+ 0·6	+ 1·8
1771	+11·8	+12·8

Modern Observations.

Year.	Observation—Theory. Hypoth. I.	Hypoth. II.
	″	″
1780	+0·27	+0·54
1783	−0·23	−0·21
1786	−0·96	−1·10
1789	+1·82	+1·63
1792	−0·91	−1·06
1795	+0·09	+0·04
1798	−0·99	−0·93
1801	−0·04	+0·11
1804	+1·76	+1·94
1807	−0·21	−0·08
1810	+0·56	+0·61
1813	−0·94	−1·00
1816	−0·31	−0·46
1819	−2·00	−2·19
1822	+0·30	+0·14
1825	+1·92	+1·87
1828	+2·25	+2·35
1831	−1·06	−0·82
1834	−1·44	−1·17
1837	−1·62	−1·53
1840	+1·73	+1·31

The largest difference in the above table, *viz.*, that for 1771, is deduced from a single observation; whereas the difference immediately preceding it, which is deduced from the mean of several, is very small.

53. The results of the two theories agree very closely with each other, and with observation, till we come to the later years of the series; and it is to be observed that the difference between the theories becomes sensible at precisely the point where they both show symptoms of diverging from the observations, the errors of the second hypothesis, however, being less than those of the other.

Recent observations show that the errors of the theory soon become very sensible, though decidedly less for the second hypothesis than for the first. The following are the differences of mean longitude as deduced from theory and observation, for the oppositions of 1843, 1844, and 1845:

Year.	Observation—Theory. Hypoth. I.	Hypoth. II
	″	″
1843	+ 7·11	+ 5·77
1844	+ 8·79	+ 7·05
1845	+12·40	+10·18

For the observations of the last two years, I am indebted to the kindness of the Astronomer Royal. The three years nearly agree in showing that the errors of the first hypothesis are to those of the second in the ratio of 5 to 4, from which I inferred, in a letter to the Astronomer Royal, dated September 2, 1846, that the assumption of $\frac{a}{a'} = \sin 35° = 0 \cdot 574$, would probably satisfy all the observations very nearly.

54. The results which I have deduced from Professor Challis's observations of the planet, strongly confirm the inference that the mean distance should be considerably diminished. It is of course impossible to determine precisely, without actual calculation, the alteration in longitude which would be produced by such a diminution in the distance. By comparing the values of θ given by the two hypotheses, it may be seen, however, that if we took successively smaller and smaller values for the mean distance, the values found for the mean longitude in 1810 would probably go on diminishing, while at the same time the mean motion from 1810 to 1846 would rapidly increase, so that the corresponding values of the mean longitude at the present time would probably soon arrive at a minimum, and afterwards begin again to increase. This I believe to be the reason why the longitude found on the supposition of too large a value for the mean distance agrees so nearly with observation. In consequence of not making sufficient allowance for the increase in the mean motion, I hastily inferred, in my letter to the Astronomer Royal mentioned above, that the effect of a diminution in the mean distance would be to diminish the mean longitude.

55. I have already mentioned that I thought it unsafe to employ Flamsteed's observation of 1690 in forming the equations of condition, as the interval between it and all the others is so large. The difference between it and the theory appears to be very considerable, and greater for the second hypothesis than for the first, the errors being $+44'' \cdot 5$ and $+50'' \cdot 0$ respectively. These errors would probably be increased by diminishing the mean distance. It would be desirable that Flamsteed's manuscripts should be examined with reference to this point.

56. The corrections of the Tabular Radius Vector of Uranus may be easily deduced from those of the mean longitude by means of the following formula:

$$\begin{aligned}\frac{\delta r}{r} = {} & \frac{1}{r}\frac{dr}{d\epsilon}\delta\zeta - \frac{1}{2n}\frac{d}{dt}\delta\zeta + \frac{1}{4}\frac{\delta a}{a} - \frac{1}{2}\frac{e\,\delta e}{1-e^2} - \frac{1}{6}m' a^2 \frac{d\Lambda_0}{da} \\ & + \frac{m'}{2}\Sigma \mathrm{C}_i \cos i\,\{nt-n't+\epsilon-\epsilon'\} \\ & + m'e\,\Sigma \mathrm{D}_i \cos\,\{i\,(nt-n't+\epsilon-\epsilon')-nt-\epsilon+\varpi\} \\ & + m'e'\Sigma \mathrm{E}_i \cos\,\{i\,(nt-n't+\epsilon-\epsilon')-nt-\epsilon+\varpi'\}\end{aligned}$$

where $\delta\zeta$ denotes the whole correction of the mean longitude at the time t,

$$\frac{1}{r}\frac{dr}{d\epsilon} = e \sin\{nt+\epsilon-\varpi\} + \frac{3e^2}{2}\sin 2\{nt+\epsilon-\varpi\} \text{ nearly,}$$

$$\mathrm{C}_i = \frac{1}{2}\frac{n}{n-n'}\,a\,\mathrm{A}^i$$

$$D_i = -\frac{1}{4}\,\frac{in}{i\,(n-n')-n}\left\{2\,i\,a\,A_i + a^2\,\frac{d\,A_i}{d\,a}\right\}$$

$$E_i = \quad \frac{1}{4}\,\frac{(i-1)\,n}{i\,(n-n')-n}\left\{(2\,i-1)\,a\,A_{i-1} + a^2\,\frac{d\,A_{i-1}}{da}\right\}$$

i assuming all integral values positive and negative not including zero.

57. By substituting in this formula the values of m', δa, δe, &c., already obtained, and putting $a = 19\cdot191$, we find the following results corresponding to the two assumed values of the mean distance.

HYPOTHESIS I.

$$\frac{a}{r}\,\delta r = \frac{a}{r}\,\frac{dr}{d\varepsilon}\,\delta\zeta - \frac{a}{2}\,\frac{d\,\delta\zeta}{ndt} - 0\cdot000089$$
$$+0\cdot000069 \cos \{nt-n't+\varepsilon-\varepsilon'\}$$
$$+0\cdot000259 \cos 2\,\{nt-n't+\varepsilon-\varepsilon'\}$$
$$+0\cdot000109 \cos 3\,\{nt-n't+\varepsilon-\varepsilon'\}$$
$$+0\cdot000016 \cos \{n't+\varepsilon'-\varpi\}$$
$$-0\cdot000168 \cos \{nt-2n't+\varepsilon-2\varepsilon'+\varpi\}$$
$$+0\cdot000078 \cos \{nt-2n't+\varepsilon-2\varepsilon'+\varpi'\}$$
$$-0\cdot000049 \cos \{2nt-3n't+2\varepsilon-3\varepsilon'+\varpi\}$$
$$+0\cdot000209 \cos \{2nt-3n't+2\varepsilon-3\varepsilon'+\varpi'\}$$

HYPOTHESIS II.

$$\frac{a}{r}\,\delta r = \frac{a}{r}\,\frac{dr}{d\varepsilon}\,\delta\zeta - \frac{a}{2}\,\frac{d\,\delta\zeta}{ndt} - 0\cdot000144$$
$$+0\cdot000073 \cos \{nt-n't+\varepsilon-\varepsilon'\}$$
$$+0\cdot000266 \cos 2\,\{nt-n't+\varepsilon-\varepsilon'\}$$
$$+0\cdot000115 \cos 3\,\{nt-n't+\varepsilon-\varepsilon'\}$$
$$+0\cdot000016 \cos \{n't+\varepsilon'-\varpi\}$$
$$-0\cdot000188 \cos \{nt-2n't+\varepsilon-2\varepsilon'+\varpi\}$$
$$+0\cdot000068 \cos \{nt-2n't+\varepsilon-2\varepsilon'+\varpi'\}$$
$$-0\cdot000053 \cos \{2nt-3n't+2\varepsilon-3\varepsilon'+\varpi\}$$
$$+0\cdot000165 \cos \{2nt-3n't+2\varepsilon-3\varepsilon'+\varpi'\}$$

58. The values of $\delta\zeta$ and $\frac{d\,\delta\zeta}{dt}$ for several late years, are the following:

HYPOTHESIS I.

Year.	$\delta\zeta$	$\frac{d\,\delta\zeta}{dt}$
	″	″
1834	− 21 ·19	−20 ·93
1840	− 74 ·82	−32 ·34
1846	−148 ·65	−39 ·94

HYPOTHESIS II.

	″	″
1834	− 21 ·46	−20 ·85
1840	− 74 ·40	−31 ·62
1846	−145 ·91	−38 ·30

Hence, by means of the above formulæ, we find the corrections of the tabular radius vector, to be

Year.	Hypothesis I.	Hypothesis II.
1834	+0 ·00505	+0 ·00492
1840	+0 ·00722	+0 ·00696
1846	+0 ·00868	+0 ·00825

59. By far the most important part of these corrections arises from the term $-\frac{1}{2}r\frac{d\delta\zeta}{ndt}$, and may therefore be immediately deduced from a comparison of the observed angular motion of Uranus with that given by the Tables. In fact, the corrections given by this term alone for the epochs above-mentioned, are

Year.	Hypothesis I.	Hypothesis II.
1834	+0 ·00447	+0 ·00445
1840	+0 ·00694	+0 ·00678
1846	+0 ·00853	+0 ·00818

which, as we see, differ very little from the complete values just found. The correction for 1834, very nearly agrees with that which Mr. Airy has deduced from observation in the *Astronomische Nachrichten*. The corrections for subsequent years are rather larger than those given by the Greenwich Observations, the results of the second hypothesis being, as in the case of the longitude, nearer the truth than those of the first.

60. I made some attempts, by discussing the observations of latitude, to find approximate values of the longitude of the node and inclination of the orbit of the disturbing planet, but the results were not satisfactory. The perturbations of the latitude are in fact exceedingly small, and during the comparatively short period of three-fourths of a revolution, are nearly confounded with the effects of a constant alteration in the inclination and the position of the node of URANUS, so that very small errors in the observations may entirely vitiate the result.

61. The perturbations of Saturn produced by the new planet, though small, will still be sensible, and it would be interesting to inquire whether, if they were taken into account, the values of the masses of Jupiter and Uranus found from their action on Saturn would be more consistent with those determined by other means, than they appear to be at present. The reduction of the Greenwich Planetary Observations renders such an inquiry comparatively easy, and it is to be hoped that English astronomers will not be the last to avail themselves of the treasures of observation thus laid open to the world.

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